LECTURE NOTES FOR MATH 327 - TOPICS IN COMPLEX ANALYSIS

0. Recapitulation of some results in complex analysis

In this section we recall some standard theorems from complex analysis that we will use in the course. We give no proofs, as they can be found in (almost) any textbook on complex analysis.

Unless stated otherwise we shall use the following notations:

- $\mathbb{C}$: complex numbers
- U: an open subset of $\mathbb{C}$
- D: a domain (open and path-connected subset of $\mathbb{C}$)
- $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$: the open ball of radius $r > 0$ and center $z_0 \in \mathbb{C}$

Definition 0.1. A function $f : U \to \mathbb{C}$ is called complex differentiable at $z_0 \in U$ if there exists the limit

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(z_0 + h) - f(z_0)}{h} \in \mathbb{C}.$$

It is called holomorphic on U if it is complex differentiable at every $z_0 \in U$.

Theorem 0.2 (Cauchy's integral formula)**.** *Let $f : U \to \mathbb{C}$ be holomorphic and suppose that the closed disc $\overline{B_r(z_0)}$ is contained in U. Then for every $a \in B_r(z_0)$ we have*

$$f(a) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{z - a}\, dz,$$

where the circle $\partial B_r(z_0)$ is oriented counterclockwise.

Corollary 0.3 (Analyticity of holomorphic functions)**.** *Under the assumptions of Theorem 0.2 the function f is analytic on U and each $f^{(k)} : U \to \mathbb{C}$ is holomorphic with*

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{(z - a)^{k+1}}\, dz.$$

Corollary 0.4 (Liouville's theorem)**.** *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Theorem 0.5 (Morera's theorem)**.** *Let $f : U \to \mathbb{C}$ be continuous. If for each triangle $\Delta \subset U$ it holds that*

$$\int_{\partial \Delta} f(z)\, dz = 0,$$

then f is holomorphic on U.

Theorem 0.6 (Identity theorem)**.** *Let $D \subset \mathbb{C}$ be a domain and $f, g : D \to \mathbb{C}$ be holomorphic. If the set $\{z \in \mathbb{C} : f(z) = g(z)\}$ has an accumulation point in D, then $f = g$.*

Theorem 0.7 (Open mapping theorem)**.** *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be a non-constant holomorphic function. Then $f(D)$ is a domain as well.*

Corollary 0.8 (Maximum principle)**.** *Let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be a holomorphic function. If $|f|$ attains its maximum on D then f is constant.*

THE FIRST VERSION OF THESE NOTES WAS KINDLY PROVIDED BY MATTHIAS RUF.

Singularities of holomorphic functions. Isolated singularities of holomorphic functions are subdivided into three categories.

Definition 0.9. Let $U \subset \mathbb{C}$ be open and let $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. Then z_0 is called

(i) a removable singularity if f can be extended to a holomorphic function $\tilde{f} : U \to \mathbb{C}$;
(ii) a pole if there exists $m \in \mathbb{N}$ such that $z \mapsto (z - z_0)^m f(z)$ has a removable singularity in z_0. The smallest such m is called the order of the pole;
(iii) an essential singularity if z_0 is neither a removable singularity nor a pole.

Theorem 0.10 (Laurent series expansion)**.** *Let $0 \le r < R$ and let $f : \{z \in \mathbb{C} : r < |z - z_0| < R\} \to \mathbb{C}$ be holomorphic. Then f has the representation*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where, for all $n \in \mathbb{Z}$ and $r < \rho < R$,

$$c_n = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

The term $\sum_{n=-\infty}^{-1} c_n (z - z_0)^n$ is called the principal part of f, while the term $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ is called the regular (or holomorphic) part of f.

Corollary 0.11. *Let $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then z_0 is*

(i) a removable singularity $\iff$ $c_k = 0 \quad \forall k < 0$ $\iff$ f is bounded near z_0;
(ii) a pole of order m $\iff$ $c_k = 0 \quad \forall k < -m$ and $c_{-m} \neq 0$;
(iii) an essential singularity $\iff$ $c_k \neq 0$ for infinitely many $k < 0$.

1. Sequences of holomorphic functions

Next we consider sequences of holomorphic functions $f_n : U \to \mathbb{C}$ and their convergence properties, i.e., compactness, convergence criteria and properties of the limit. As we shall see the following notion of convergence is well-suited with regard to the above properties.

Definition 1.1. A sequence $f_n : U \to \mathbb{C}$ of holomorphic functions is said to converge locally uniformly to some function $f : U \to \mathbb{C}$ if for each $z_0 \in U$ there exists $r > 0$ such that

$$\sup_{z \in B_r(z_0)} |f_n(z) - f(z)| \to 0 \quad \text{as } n \to \infty.$$

Remark 1.2. Local uniform convergence is equivalent to uniform convergence on each compact subset of U.

The following theorem shows that local uniform convergence preserves holomorphy.

Theorem 1.3. *Assume that a sequence $f_n : U \to \mathbb{C}$ of holomorphic functions converges locally uniformly to some $f : U \to \mathbb{C}$. Then f is holomorphic.*

Proof. Note that f is continuous as the locally uniform limit of continuous functions. Hence by Morera's theorem it suffices to check that for each triangle $\Delta \subset U$ we have

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

Since $f_n \to f$ uniformly on Δ by Remark 1.2 we conclude from Cauchy's theorem that

$$0 = \lim_{n\to\infty} \int_{\partial\Delta} f_n(z)\, dz = \int_{\partial\Delta} f(z)\, dz,$$

where the last equality can be justified for instance by Lebesgue's dominated convergence theorem. This proves the claim.

□

Remark 1.4. Theorem 1.3 is in general false for pointwise converging sequences of holomorphic functions (an example can be found in [1]). However, Osgood's theorem [4, p. 151] (see also exercise H 2.4) ensures that the pointwise limit is holomorphic on a dense, open subset of U.

For sequences $f_n : \mathbb{R} \to \mathbb{R}$ uniform convergence does not allow to conclude convergence of the derivatives. For instance, the sequence $f_n(x) = \frac{1}{n}\sin(nx)$ converges uniformly to 0, but its derivative $f_n'(x) = \cos(nx)$ does not even converge pointwise. As we prove next, holomorphic functions behave much better.

Theorem 1.5. *Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly to $f : U \to \mathbb{C}$. Then for each $k \in \mathbb{N}$ the sequence $f_n^{(k)}$ converges locally uniformly to $f^{(k)}$.*

Proof. Let $z_0 \in U$ and $r > 0$ be such that $\overline{B_{2r}(z_0)} \subset U$. Due to Cauchy's integral formula, for all $z' \in B_r(z_0)$ we can write

$$f^{(k)}(z') - f_n^{(k)}(z') = \frac{k!}{2\pi i}\int_{\partial B_{2r}(z_0)} \frac{f(z) - f_n(z)}{(z - z')^{k+1}}\, dz.$$

Note that for $z' \in B_r(z_0)$ and $z \in \partial B_{2r}(z_0)$ it holds that $|z - z'| \geq r$. Since the length of $\partial B_{2r}(z_0)$ equals $4\pi r$ we deduce that

$$\sup_{z' \in B_r(z_0)} |f^{(k)}(z') - f_n^{(k)}(z')| \leq \frac{2k!}{r^k} \sup_{z' \in B_{2r}(z_0)} |f(z') - f_n(z')|.$$

Due to Remark 1.2 the right hand side converges to 0 when $n \to \infty$ and we conclude the proof. □

The previous theorem allows to control the number of zeros of the limit of holomorphic functions.

Corollary 1.6. *Let $D \subset \mathbb{C}$ be a domain and $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly to $f : D \to \mathbb{C}$. If each f_n has at most m zeros (counted with multiplicity), then either $f \equiv 0$ or f has at most m zeros.*

Proof. Let $f \not\equiv 0$ and assume by contradiction that f has distinct zeros $z_1, \ldots, z_\ell$ with total multiplicity larger than m. By the identity theorem the zeros of f are isolated, so that for each z_j we find a ball $B_r(z_j)$ such that

(i) $\{f = 0\} \cap \overline{B_r(z_j)} = \{z_j\}$,

(ii) $\overline{B_r(z_j)} \cap \overline{B_r(z_i)} = \emptyset \quad \forall 1 \leq j \neq i \leq \ell$.

The argument principle then implies

$$m + 1 \leq \sum_{j=1}^{\ell} \frac{1}{2\pi i}\int_{\partial B_r(z_j)} \frac{f'(z)}{f(z)} = \sum_{j=1}^{\ell} \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B_r(z_j)} \frac{f_n'(z)}{f_n(z)}\, dz \leq m,$$

where in the second equality we used dominated convergence, Theorem 1.5, and that $f \neq 0$ on the compact set $\partial B_r(z_j)$, so for $n \geq n_\varepsilon$ we have the uniform bound $|f_n| > \varepsilon$ on the same set. This yields a contradiction. □

Next we turn our attention to convergence criteria. The first one is a general compactness result.

Theorem 1.7 (Montel's theorem)**.** *Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded, i.e., for each $z_0 \in U$ there exists $r > 0$ and $C < \infty$ such that*

$$\sup_{n \in \mathbb{N}} \sup_{z \in B_r(z_0)} |f_n(z)| \leq C.$$

Then there exists a subsequence f_{n_k} that converges locally uniformly to a holomorphic function $f : U \to \mathbb{C}$.

Proof. Take a countable, dense subset S of U (e.g. $(\mathbb{Q}+i\mathbb{Q})\cap U$) and let us write $S=\{z_1,z_2,z_3,\dots\}$. Since the sequence $\{f_n(z_1)\}_{n\in\mathbb{N}}$ is bounded, we may apply the Bolzano-Weierstrass theorem in order to extract a subsequence $n_{k,1}$ such that $f_{n_{k,1}}(z_1)$ converges to some value $f_{z_1}\in\mathbb{C}$. In a next step we note that the sequence $\{f_{n_{k,1}}(z_2)\}_{k\in\mathbb{N}}$ is again bounded, so that by the same reasoning we find another subsequence $n_{k,2}$ of the previous subsequence such that $\{f_{n_{k,2}}(z_2)\}_{k\in\mathbb{N}}$ converges to some value $f_{z_2}\in\mathbb{C}$. In the j^{th} step we choose a subsequence $n_{k,j}$ of all previous subsequences such that $\{f_{n_{k,j}}\}_{k\in\mathbb{N}}$ converges to some value $f_{z_j}\in\mathbb{C}$. For $k\in\mathbb{N}$ we finally set $n_k:=n_{k,k}$. Then the sequence $f_{n_k}(z_j)$ converges to f_{z_j} for all $j\in\mathbb{N}$ since except for finitely many terms the sequence n_k is a subsequence of $\{n_{k,j}\}_{k\in\mathbb{N}}$. Thus we found a subsequence f_{n_k} such that $f_{n_k}(z_j)$ converges to some value $f_{z_j}\in\mathbb{C}$, for every $z_j\in S$.

Next we show that f_n is equicontinuous (i.e. that the $\varepsilon-\delta$ definition of continuity is valid with δ independent of n). Fix $z_0\in U$ and let $r>0$ be such that $\overline{B_{2r}(z_0)}\subset U$ and such that there exists $C<\infty$ with

$$\sup_{n\in\mathbb{N}}\sup_{z\in\overline{B_{2r}(z_0)}}|f_n(z)|\le C.$$

By Cauchy's integral formula, for all $z'\in B_r(z_0)$ we have

$$\begin{aligned}|f_n(z')-f_n(z_0)|=&\left|\frac{1}{2\pi i}\int_{\partial B_{2r}(z_0)}\frac{f_n(z)}{z-z'}-\frac{f_n(z)}{z-z_0}\,dz\right|=\frac{|z'-z_0|}{2\pi}\left|\int_{\partial B_{2r}(z_0)}\frac{f_n(z)}{(z-z')(z-z_0)}\,dz\right|\\ \le&\frac{|z'-z_0|}{2\pi}\frac{C\cdot 4\pi r}{2r^2}=\frac{C|z'-z_0|}{r},\end{aligned}$$

where we used that $|z-z'|\ge r$ and $|z-z_0|\ge 2r$ for all $z\in\partial B_{2r}(z_0)$. The right hand side is independent of n, so given $\varepsilon>0$ we can choose $\delta_{\varepsilon,z_0}=\min\{r,\varepsilon\frac{r}{C}\}$ in the definition of continuity. Hence f_n is equicontinuous.

Equicontinuity allows us to show that $\{f_{n_k}(z)\}_{k\in\mathbb{N}}$ is a Cauchy sequence for all $z\in U$. To reduce notation, we skip the subscript k. For $z\in U$ and $\varepsilon>0$ we first choose $z^*\in S$ such that $|z-z^*|<\delta_{\varepsilon,z}$, where $\delta_{\varepsilon,z}$ satisfies the equicontinuity condition

$$|y-z|<\delta_{\varepsilon,z}\quad\Rightarrow\quad|f_n(y)-f_n(z)|<\frac{\varepsilon}{3}\qquad\forall n\in\mathbb{N}.\tag{1}$$

To find such a z^* is possible due to the density of S in U. For $m\ge n$ we then have

$$|f_m(z)-f_n(z)|\le\underbrace{|f_m(z)-f_m(z^*)|}_{<\varepsilon/3}+|f_m(z^*)-f_n(z^*)|+\underbrace{|f_n(z^*)-f_n(z)|}_{<\varepsilon/3}$$

Since $z^*\in S$ the convergence on S implies that there exists $n_{\varepsilon,z}\in\mathbb{N}$ such that for all $m\ge n\ge n_{\varepsilon,z}$ we have $|f_m(z^*)-f_n(z^*)|<\frac{\varepsilon}{3}$. Then for all $m\ge n\ge n_{\varepsilon,z}$ we conclude that

$$|f_m(z)-f_n(z)|<\varepsilon.$$

Hence $\{f_n(z)\}_{n\in\mathbb{N}}$ is Cauchy sequence as claimed, so that there exists $f_z=\lim_{n\to\infty}f_n(z)$ for all $z\in U$.

Finally, we show that f_n converges locally uniformly to $f(z):=f_z$. Fix a compact set $K\subset U$. Given $\varepsilon>0$ and $z\in K$ we choose $\delta_{\varepsilon,z}>0$ satisfying (1) above. Then the family of discs $\{B_{\delta_{\varepsilon,z}}(z)\}_{z\in K}$ forms an open cover of K. By the (topological) definition of compactness there exists a finite sub-family $\{B_{\delta_{\varepsilon,z_i}}\}_{i=1}^N$ with $z_i\in K$ that still covers K. Thus for any $z\in K$ we find z_i such that $|z-z_i|<\delta_{\varepsilon,z_i}$. Since the $\{z_i\}$ are only finitely many there exists $n_\varepsilon\in\mathbb{N}$ such that for all $n\ge n_\varepsilon$ it holds that

$$|f(z_i)-f_n(z_i)|<\frac{\varepsilon}{3}.$$

Moreover, observe that (1) also holds for the limit function f as we can pass to the limit in this estimate. Consequently, for $n\ge n_\varepsilon$ we deduce that for all $z\in K$ we have

$$|f(z)-f_n(z)|\le\underbrace{|f(z)-f(z_i)|}_{\le\varepsilon/3}+\underbrace{|f(z_i)-f_n(z_i)|}_{<\varepsilon/3}+\underbrace{|f_n(z_i)-f_n(z)|}_{<\varepsilon/3}<\varepsilon,$$

which shows the uniform convergence of f_n to f and we conclude the proof.

□

Finally, we state two criteria which ensure the convergence along the whole sequence.

Theorem 1.8 (Vitali's theorem)**.** *Let $D \subset \mathbb{C}$ be a domain and let $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded. If the set $L := \{z \in D : \lim_{n\to\infty} f_n(z) \text{ exists}\}$ has an accumulation point in D, then f_n converges locally uniformly to some holomorphic function $f : D \to \mathbb{C}$.*

Proof. Due to Montel's theorem there exists a subsequence f_{n_k} that converges locally uniformly to a holomorphic function $f : D \to \mathbb{C}$. Note that local uniform convergence is induced by a topology. Hence the non-convergence of the whole sequence to f implies that there exists another subsequence which has no subsequence that does converge locally uniformly to f. Applying Montel's theorem along this subsequence, we obtain another subsequence $f_{n_{k,1}}$ that converges locally uniformly to a holomorphic function $h : D \to \mathbb{C}$. Then $h \neq f$, but $h(z) = f(z)$ for all $z \in L$, which contradicts the identity theorem. □

Theorem 1.9. *Let $D \subset \mathbb{C}$ be a domain and let $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded. If for all $k \in \mathbb{N} \cup \{0\}$ and some $z_0 \in D$ the sequences $f_n^{(k)}(z_0)$ converge, then f_n converges locally uniformly to some holomorphic function $f : D \to \mathbb{C}$.*

Proof. See Exercise H 3.1. □

Local normal convergence. In the next chapter the focus will be on series of holomorphic functions. For those the following concept of convergence will be useful.

Definition 1.10. Let $f_j : U \to \mathbb{C}$ be a sequence of complex-valued functions. The series $\sum_{j=1}^{\infty} f_j$ is called locally normally convergent if for each $z_0 \in U$ there exists $r > 0$ such that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)| < \infty.$$

As shown in the lemma below, local normal convergence implies local uniform convergence. In the exercises we will see that the converse is false in general.

Lemma 1.11. *Let $f_j : U \to \mathbb{C}$ be a sequence of complex-valued functions. If the series $\sum_{j=1}^{\infty} f_j$ converges locally normally, then it also converges locally uniformly. In particular, if each f_j is in addition holomorphic, then $z \mapsto \sum_{j=1}^{\infty} f_j(z)$ is also holomorphic.*

Proof. See Exercise H 3.4 and Theorem 1.3. □

2. The Mittag-Leffler theorem

We start with the following simple observation: if $\{d_1, \dots, d_n\} \subset \mathbb{C}$ is a finite set and for each d_n the function $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ denotes a finite principal part at d_n given by $q_n(z) = \sum_{j=1}^{m_n} a_{nj}(z - d_n)^{-j}$, then the function

$$f(z) = \sum_{n=1}^{N} q_n(z)$$

is meromorphic on $\mathbb{C}$ and at each $d_n \in \mathbb{C}$ the principal part of its Laurent series agrees with q_n. In 1876/77 the Swedish mathematician Gösta Mittag-Leffler extended the above result to closed discrete sets $\{d_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ (i.e. sets with no accumulation points). In 1880 Karl Weierstraß found a simplified proof which in general also allows for $m_n = \infty$ (albeit with some implicit growth conditions on the coefficients a_{nj} by requiring that $q_n : \mathbb{C} \setminus \{d_n\}$ is holomorphic). In this course we shall follow the argument of Weierstrass, but prove a more general version valid on open sets. To reduce notation, we introduce some vocabulary.

Definition 2.1. Let $d \in \mathbb{C}$ and $q : \mathbb{C} \setminus \{d\} \to \mathbb{C}$ be a holomorphic function. Then q is called a principal part at d when its Laurent series expansion around d has no regular part.

With this definition the theorem of Mittag-Leffler on $\mathbb{C}$ reads as follows:

Theorem 2.2 (Mittag-Leffler on $\mathbb{C}$). *Let $S = \{d_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a set with no accumulation points. For each $d_n \in S$ let $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ be a principal part. Then there exists a holomorphic function $f : \mathbb{C}\setminus S \to \mathbb{C}$ such that at each $d_n \in S$ the principal part of its Laurent series is given by q_n. The function f can be taken to be of the form*

$$f(z) = \sum_{n=1}^{\infty} q_n(z) - p_n(z),$$

where $p_n : \mathbb{C} \to \mathbb{C}$ is a polynomial and the sum converges locally normally on $\mathbb{C} \setminus S$.

Proof. Without loss of generality we may assume that $0 < |d_1| \le |d_2| \le \dots$ (if $d_1 = 0$ then set $p_1 \equiv 0$ and separate this term from the analysis). One cannot expect that the series $\sum_{n=1}^{\infty} q_n$ converges, so we have to use the polynomials to ensure convergence. As they should be close to q_n, we take suitable Taylor polynomials. Since q_n is holomorphic on $B_{|d_n|}(0)$, it has a convergent series expansion on this ball. That is, for any $z \in B_{|d_n|}(0)$ it holds that

$$q_n(z) = \sum_{j=0}^{\infty} a_{nj}\, z^j.$$

By general properties of power series we know that on the smaller ball $B_{\frac{|d_n|}{2}}(0)$ the above series converges uniformly. Hence for each $n \in \mathbb{N}$ there exists a number $j_n \in \mathbb{N}$ such that

$$\sup_{z\in B_{\frac{|d_n|}{2}}(0)} \Big| q_n(z) - \underbrace{\sum_{j=0}^{j_n} a_{nj}\, z^j}_{=:p_n(z)} \Big| \le 2^{-n}. \tag{2}$$

Next fix a compact set $K \subset \mathbb{C}\setminus S$. Since S has no accumulation points, we know that $\lim_n |d_n| = \infty$. As K is in particular bounded, we find a number $n(K) \in \mathbb{N}$ such that for all $n \ge n(K)$ we have $K \subset B_{\frac{|d_n|}{2}}(0)$. Consequently, from (2) we infer that

$$\sum_{n\ge n(K)} \sup_{z\in K} |q_n(z) - p_n(z)| \le \sum_{n\ge n(K)} \sup_{z\in B_{\frac{|d_n|}{2}}(0)} |q_n(z) - p_n(z)| \le \sum_{n\ge n(K)} 2^{-n} < \infty.$$

Since K is a compact subset of $\mathbb{C} \setminus S$, each function $q_n - p_n$ is bounded on K. Using this for $n < n(K)$, we have shown that the series

$$f := \sum_{n=1}^{\infty} q_n - p_n$$

converges locally normally on $\mathbb{C} \setminus S$. In particular, by Lemma 1.11 it is holomorphic. Finally, in order to obtain the principal part at a point d_n we argue as follows: choose $\rho > 0$ such that $B_{2\rho}(d_n) \cap S = \{d_n\}$. Then by the formula for the Laurent coefficients and local uniform convergence of f we deduce that the j-th Laurent coefficient at d_n, denoted here by $a_j(d_n)$, is given by

$$a_j(d_n) = \frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{f(z)}{(z-d_n)^{j+1}}\, dz = \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{q_k(z) - p_k(z)}{(z-d_n)^{j+1}}\, dz.$$

For $j \le -1$ the only integrand that is not holomorphic on $B_{2\rho}(d_n)$ is $q_n(z)(z-d_n)^{-(j+1)}$. All other contributions vanish due to Cauchy's integral theorem. Hence for $j \le -1$ we deduce that

$$a_j(d_n) = \frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{q_n(z)}{(z-d_n)^{j+1}}\, dz,$$

which coincides with the j-th Laurent coefficient at d_n of q_n. Thus the principal part at d_n is given by q_n, as claimed.

□

Remark 2.3. Any other holomorphic function $\tilde{f} : \mathbb{C} \setminus S \to \mathbb{C}$ with the same principal parts at $d_n \in S$ differs from f by an entire function. Indeed, the difference $f - \tilde{f}$ has removable singularities at each $d_n \in S$ since all its Laurent coefficients with negative index vanish.

Now we prepare to extend the previous theorem to general open sets $U \subset \mathbb{C}$ and point sets $S = \{d_n\}_{n\in\mathbb{N}} \subset U$ which have no accumulation points in U (but may have accumulation points at ∂U). Instead of polynomials we will use truncated Laurent series to ensure convergence of the series.

Definition 2.4. Given a Laurent series $q(z) = \sum_{j=1}^{\infty} a_{-j}(z-d)^{-j}$ and $k \in \mathbb{N}$, we define the truncated Laurent series $q^k(z) = \sum_{j=1}^{k} a_{-j}(z-d)^{-j}$.

Before proceding to the general Mittag-Leffler theorem, we need some auxiliary results concerning principal parts.

Lemma 2.5. *Let $q : \mathbb{C} \setminus \{d\} \to \mathbb{C}$ be holomorphic.*

(i) q is a principal part $\iff$ $\lim_{|z|\to\infty} q(z) = 0$.

(ii) If q is a principal part, then for any $c \in \mathbb{C}$ the function q admits a Laurent series expansion on the annulus $A_c = \{z \in \mathbb{C} : |z-c| > |d-c|\}$ of the form

$$q(z) = \sum_{j=1}^{\infty} \tilde{a}_{-j}(z-c)^{-j}.$$

Proof. (i) By Theorem 0.10 we can write $q = q^+ + q^-$, where the regular part $q^+ : \mathbb{C} \to \mathbb{C}$ is an entire function and the principal part $q^- : \mathbb{C} \setminus \{d\} \to \mathbb{C}$ is of the form $q^-(z) = \sum_{j=1}^{\infty} a_{-j}(z-d)^{-j}$. First we recall a general bound on the coefficients of a Laurent series expansion. In our case, for all $\rho > 0$ and $j \in \mathbb{Z}$ it holds that

$$|a_j| = \left| \frac{1}{2\pi i} \int_{\partial B_\rho(d)} \frac{q(z)}{(z-d)^{j+1}} \right| \le \frac{1}{2\pi} \underbrace{\text{Length}(\partial B_\rho(d))}_{=2\pi\rho} \frac{\sup_{z\in\partial B_\rho(d)} |q(z)|}{\rho^{j+1}} = \sup_{z\in\partial B_\rho(d)} \frac{|q(z)|}{\rho^j}. \tag{3}$$

Inserting $\rho = 1$ we see that $|a_j| \le C$ for some constant independent of j. From that it is not difficult to prove that

$$\lim_{|z|\to\infty} q^-(z) = 0. \tag{4}$$

If q is a principal part, then $q^+ \equiv 0$ so that (4) implies that $\lim_{|z|\to\infty} q(z) = 0$. On the other hand, if $\lim_{|z|\to\infty} q(z) = 0$ then (4) yields that $\lim_{|z|\to\infty} q^+(z) = 0$. Since q^+ is an entire function we deduce from Liouville's theorem that $q^+ \equiv 0$, so that q is a principal part. This proves the equivalence (i).

(ii) In order to prove the second statement, note that since q is holomorphic on $A_c \subset \mathbb{C} \setminus \{d\}$, it admits a Laurent series representation $q(z) = \sum_{j=-\infty}^{\infty} \tilde{a}_j (z-c)^j$ for $z \in A_c$. Thus we just need to show that $\tilde{a}_j = 0$ for all $j \ge 0$. Arguing as in (3), for any $\rho > |d-c|$ the coefficients satisfy the estimate

$$|\tilde{a}_j| \le \sup_{z\in\partial B_\rho(c)} \frac{|q(z)|}{\rho^j}.$$

For $\rho \ge 1$ we have $\rho^{-j} \le 1$ for all $j \ge 0$. By (i) we know that the supremum vanishes when $\rho \to \infty$. Hence $\tilde{a}_j = 0$ for all $j \ge 0$, as claimed.

□

Now we can prove the general Mittag-Leffler theorem for a special class of sets S. First some notation. Given a set $S \subset U$ with no accumulation points in U, we define $S' = \overline{S} \setminus S$, the set of its accumulation points in $\mathbb{C}$.

Proposition 2.6. *Let $S = \{d_n\}_{n\in\mathbb{N}} \subset U$ be a set with no accumulation points in U. For each $d_n \in S$ let $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ be a principal part. If there exists a sequence $\{c_n\}_{n\in\mathbb{N}} \subset S'$ such that $\lim_n |d_n - c_n| = 0$*

then there exist truncated principal parts $q_n^{k_n}$ *centered in* c_n *(cf. Lemma 2.5(ii)) such that the series*

$$f = \sum_{n=1}^{\infty} q_n - q_n^{k_n}$$

converges locally normally in $\mathbb{C} \setminus \overline{S} \supset U \setminus S$ *and at each point* $d_n \in S$ *the principal part of* f *is given by* q_n.

Proof. Note that by Lemma 2.5(ii) and general properties of Laurent series (i.e. bounds on coefficients), the sequence of truncated Laurent series q_n^k centered at c_n converges uniformly to q_n on the smaller annulus $A_{c_n}^2 := \{z \in \mathbb{C} : |z - c_n| \geq 2|d_n - c_n|\}$ as $k \to \infty$. Hence for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that

$$|q_n(z) - q_n^{k_n}(z)| \leq 2^{-n} \quad \forall z \in A_{c_n}^2. \tag{5}$$

Now fix a compact subset K of $\mathbb{C} \setminus \overline{S}$. Then $\operatorname{dist}(\overline{S}, K) > \varepsilon$ for some $\varepsilon > 0$ and since $\lim_n |d_n - c_n| = 0$ we find an index $n(K) \in \mathbb{N}$ such that for all $n \geq n(K)$ it holds that

$$K \subset A_{c_n}^2 = \{z \in \mathbb{C} : \underbrace{|z - c_n|}_{\geq \varepsilon \text{ on } K} \geq \underbrace{2|d_n - c_n|}_{\to 0}\},$$

where we recall that $c_n \in S' \subset \overline{S}$. Consequently we can use (5) to estimate

$$\sum_{n \geq n(K)} \sup_{z \in K} |q_n(z) - q_n^{k_n}(z)| \leq \sum_{n \geq n(K)} 2^{-n} < \infty.$$

Since the singularities of q_n and $q_n^{k_n}$ are contained in $\overline{S}$, each function $q_n - q_n^{k_n}$ is bounded on K. Hence we have shown the local normal convergence of

$$f = \sum_{n=1}^{\infty} q_n - q_n^{k_n}.$$

In particular, f is holomorphic on $\mathbb{C} \setminus \overline{S}$. Finally, since each $q_n^{k_n}$ is holomorphic on U (recall that we have chosen the centers c_n of the truncated Laurent series in $S' \subset \partial U$), by the same reasoning as for Theorem 2.2 we conclude that at each $d_n \in S$ the principal part of f is given by q_n.

□

As a next step we divide the set of singularities S in a suitable way so that for one subset we can apply the Mittag-Leffler theorem on $\mathbb{C}$, and for the other set we can apply the special version above. The basic idea is to split S into a closed set and sets close to an accumulation point. The following lemma makes this splitting precise.

Lemma 2.7. *Let* $S \subset U$ *be a set with no accumulation points in* U *such that* $S' = \overline{S} \setminus S \neq \emptyset$. *Define*

$$S_1 := \{z \in S : |z| \cdot \operatorname{dist}(S', z) \geq 1\}, \qquad S_2 := \{z \in S : |z| \cdot \operatorname{dist}(S', z) < 1\}.$$

Then S_1 *is closed and for every* $\varepsilon > 0$ *the set* $S_2(\varepsilon) := \{z \in S_2 : \operatorname{dist}(S', z) \geq \varepsilon\}$ *is finite.*

Proof. We first prove that S_1 is closed. Let $\{z_n\}_{n \in \mathbb{N}} \subset S_1$ be a sequence such that $z_n \to z^*$ for some $z^* \in \mathbb{C}$. Due to the continuity of the function $z \mapsto |z| \cdot \operatorname{dist}(S', z)$ we know that $|z^*| \cdot \operatorname{dist}(S', z^*) \geq 1$. We claim that $z^* \in S$, which shows that S_1 is closed. Indeed, assume by contradictions that $z^* \notin S$. Then by definition $z^* \in S'$, which contradicts the fact that $|z^*| \cdot \operatorname{dist}(S', z^*) \geq 1$.

In order to prove the second assertion, note that for any $z \in S_2(\varepsilon)$ we have by definition

$$|z| \leq \operatorname{dist}(S', z)^{-1} \leq \frac{1}{\varepsilon}.$$

Thus, assuming by contradiction that $S_2(\varepsilon)$ is infinite for some $\varepsilon > 0$, we obtain a sequence of distinct points $\{z_n\}_{n \in \mathbb{N}} \subset S_2(\varepsilon)$ such that $z_n \to z^*$ for some $z^* \in \overline{S}$. Since $S \subset U$ does not contain any accumulation point it follows that $z^* \in S'$. But due to continuity it holds that $\operatorname{dist}(S', z^*) \geq \varepsilon$, which yields a contradiction.

□

The splitting $S = S_1 \sqcup S_2$ can be further justified by the following property which will allow us to apply Proposition 2.6.

Lemma 2.8. *Let $S_2 = \{d_n\}_{n\in\mathbb{N}} \subset U$ be as in Lemma 2.7 and assume that $S_2' \neq \emptyset$. Then there exists a sequence $\{c_n\}_{n\in\mathbb{N}} \subset S_2'$ such that $\lim_n |d_n - c_n| = 0$.*

Proof. First note that S_2' is closed (this is a general fact which can be proven by a diagonal argument, but even more direct here since S_2 is discrete, hence $S_2' = \overline{S_2} \setminus S_2$). Hence for each $n \in \mathbb{N}$ there exists $c_n \in S_2'$ such that $\operatorname{dist}(S_2', d_n) = |d_n - c_n|$. If $\operatorname{dist}(S_2', d_n)$ does not converge to zero, then for some $\varepsilon > 0$ the set $S_2(\varepsilon)$ defined in Lemma 2.7 would be infinite. Indeed, the assumption $S_2' \neq \emptyset$ implies that S_2 is infinite. Moreover, we have that $S' = S_2'$ since the set S_1 is closed (and discrete).

□

Now we can state and prove the full theorem of Mittag-Leffler on open sets, which will be the final result of this chapter.

Theorem 2.9. *Let $U \subset \mathbb{C}$ be open and let $S = \{d_n\}_{n\in\mathbb{N}} \subset U$ be a set with no accumulation points in U. For each d_n let $q_n : \mathbb{C} \setminus \{d_n\} \to \mathbb{C}$ be a principal part. Then there exists a holomorphic function $f : U \setminus S \to \mathbb{C}$ such that at each d_n its principal part is given by q_n. The function f can be taken to be of the form*

$$f = \sum_{n=1}^{\infty} q_n - h_n,$$

where each $h_n : U \to \mathbb{C}$ is holomorphic and the series converges locally normally on $U \setminus S$.

Proof. Without loss of generality we may assume that $S' = \overline{S} \setminus S \neq \emptyset$ (otherwise S has no accumulation points in $\mathbb{C}$ and we can apply Theorem 2.2). Let S_1 and S_2 be defined as in Lemma 2.7. Since S_1 is closed (by that lemma), we know that $S_1' = \emptyset$ and therefore $S_2' = S'$. Let us write $S_1 = \{d_{n,1}\}_n$ and $S_2 = \{d_{n,2}\}_n$ (we do not claim that both are infinite). Since $S_1 \subset S$ is closed and discrete in $\mathbb{C}$ we can apply Theorem 2.2 to deduce that there exists a family of polynomials $p_{n,1} : \mathbb{C} \to \mathbb{C}$ such that

$$f_1 = \sum_{d_{n,1}\in S_1} q_{n,1} - p_{n,1}$$

is holomorphic on $\mathbb{C} \setminus S_1$, at each $d_{n,1} \in S_1$ its principal part is given by $q_{n,1}$, and the series converges locally normally on $\mathbb{C} \setminus S_1$.

Next we treat the set S_2. On this set we can apply Proposition 2.6 (cf. Lemmata 2.7 & 2.8) to deduce that there exist holomorphic functions $h_{n,2} : U \to \mathbb{C}$ such that

$$f_2 = \sum_{d_{n,2}\in S_2} q_{n,2} - h_{n,2}$$

is holomorphic on $U \setminus S_2$, at each $d_{n,2} \in S_2$ its principal part is given by $q_{n,2}$, and the series converges locally normally on $U \setminus S_2$.

Since $S = S_1 \sqcup S_2$, the function $f = f_1 + f_2$ thus satisfies all the claimed properties.

□

3. Infinite products

In this chapter we deal with the counterpart of series for products. The definition of infinite products $\prod_{j\geq 1} a_j$ seems quite obvious considering the Cauchy criterion for finite partial products. However, it is customary to exclude some cases, for instance when some factors a_j equal zero or also when the limit equals zero. In this course we allow for the first case. Then the definition reads as follows:

Definition 3.1. Let $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers. The infinite product $\prod_{j=1}^{\infty} a_j$ is said to converge if there exists $j_0 \in \mathbb{N}$ such that $a_j \neq 0$ for all $j \geq j_0$ and there exists the limit

$$a(j_0) := \lim_{m\to\infty} \prod_{j=j_0}^{m} a_j \neq 0.$$

In this case we set $\prod_{j=1}^{\infty} a_j = a(j_0) \prod_{j=1}^{j_0-1} a_j$. Note that this definition is independent of the number j_0.

With the above definition an infinite product is zero if and only if some factor is zero. Moreover, we have a simple necessary condition for convergence, similar to series.

Lemma 3.2. *Assume that the infinite product $a = \prod_{j=1}^{\infty} a_j$ converges. Then for all $m \in \mathbb{N}$ the infinite product $a(m) = \prod_{j=m}^{\infty} a_j$ converges. Moreover $\lim_{m\to\infty} a(m) = 1$ and $\lim_{j\to\infty} a_j = 1$.*

Proof. Without loss of generality we may assume that $a_j \neq 0$ for all $j \in \mathbb{N}$. The convergence of the products $a(m)$ follows from the definition. Moreover we have

$$\frac{a}{a(m)} = \lim_{n\to\infty} \frac{\prod_{j=1}^{n} a_j}{\prod_{j=m}^{n} a_j} = \underbrace{\prod_{j=1}^{m-1} a_j}_{\to a \text{ as } m\to\infty} .$$

Letting $m \to \infty$, we deduce that

$$\lim_{m\to\infty} a(m) a = a.$$

Since $a \neq 0$ by definition, we deduce that $\lim_{m\to\infty} a(m) = 1$. Moreover, since $a_j = a(j)/a(j+1)$ we also conclude that $\lim_{j\to\infty} a_j = 1$.

□

Next we prove an elementary, but useful criterion for the convergence of infinite products. Without loss of generality (in light of Lemma 3.2) we shall assume that all factors are away from the non-positive real axis.

Lemma 3.3. *Let $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C} \setminus (-\infty, 0]$ be a sequence. Then $\prod_{j=1}^{\infty} a_j$ converges if and only if the series $\sum_{j=1}^{\infty} \log(a_j)$ converges, where $\log$ denotes the principal branch of the logarithm.*

Proof. First assume that $\sum_{j=1}^{\infty} \log(a_j)$ converges. Then taking the complex exponential we deduce that

$$0 \neq \exp\Big(\sum_{j=1}^{\infty} \log(a_j)\Big) = \lim_{n\to\infty} \exp\Big(\sum_{j=1}^{n} \log(a_j)\Big) = \lim_{n\to\infty} \prod_{j=1}^{n} a_j.$$

This proves the convergence of the infinite product.

To prove the reverse direction, we set $P_n = \prod_{j=1}^{n} a_j \neq 0$. It seems natural to take the logarithm of P_n. However, the equality $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ is only valid up to an additive multiple of 2π on $\mathbb{C} \setminus \{0\}$. Nevertheless, since $\prod_{j=1}^{\infty} a_j \neq 0$ we can find $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ it holds that

$$|P_n - P_m| \leq \frac{1}{2} |P_n| ,$$

or equivalently

$$\left| 1 - \frac{P_m}{P_n} \right| \leq \frac{1}{2}.$$

In particular, all the products $\prod_{j=n+1}^{m} a_j$ are contained in the right half-plane. Hence we have by the definition of the principal branch of the logarithm that

$$\log\Big(\prod_{j=n_0+1}^{m} a_j\Big) = \sum_{j=n_0+1}^{m} \log(a_j).$$

Passing to the limit as $m \to \infty$, we deduce the claim from the continuity of $z \mapsto \log(z)$ on the right half-plane.

□

For infinite products, defining absolute convergence as the convergence of $\prod_{j=1}^\infty |a_j|$ is not beneficial. On the one hand, it would not imply the convergence of $\prod_{j=1}^\infty a_j$ (for instance, take $a_j = (-1)^j$). On the other hand, the convergence of $\prod_{j=1}^\infty a_j$ always implies the convergence of $\prod_{j=1}^\infty |a_j|$ due to the property $|a \cdot b| = |a| \cdot |b|$. However, Lemma 3.3 motivates the following definition.

Definition 3.4. An infinite product $\prod_{j=1}^\infty a_j$ is called absolutely convergent if there exists $n_0 \in \mathbb{N}$ such that $a_n \notin (-\infty, 0]$ for all $n \geq n_0$ and the series $\sum_{j=n_0}^\infty \log(a_j)$ is absolutely convergent.

With this definition, absolute convergence implies convergence (by Lemma 3.3 and the corresponding result for series). Moreover, we can formulate a second useful convergence criterion.

Lemma 3.5. *An infinite product* $\displaystyle\prod_{j=1}^\infty (1 + a_j)$ *converges absolutely if and only if* $\displaystyle\sum_{j=1}^\infty |a_j|$ *converges.*

Proof. See Exercise H 5.1. □

Next we deal with infinite products of (holomorphic) functions. Given a sequence $f_j : U \to \mathbb{C}$ we distinguish two types of convergence of the product $\prod_{j=1}^\infty f_j$: local uniform and local normal convergence (cf. the corresponding notions for series).

Definition 3.6. Let $f_j : U \to \mathbb{C}$ be a sequence of continuous functions. An infinite product $\prod_{j=1}^\infty f_j : U \to \mathbb{C}$ is called locally uniformly convergent if for every $z_0 \in U$ there exist $r > 0$ and $j_0 \in \mathbb{N}$ such that $\prod_{j=j_0}^n f_j$ converges uniformly on $B_r(z_0)$ to some non-vanishing function.

It follows from the definition that a locally uniformly convergent product converges also pointwise. There are further immediate consequences that we summarize in the corollary below.

Corollary 3.7. *Let $f_j : U \to \mathbb{C}$ be a sequence of continuous functions. Assume that $f = \prod_{j=1}^\infty f_j : U \to \mathbb{C}$ converges locally uniformly. Then*

(i) the sequence $\prod_{j=n}^\infty f_j$ converges locally uniformly to 1 as $n \to \infty$. In particular, we have that $f_j \to 1$ locally uniformly as $j \to \infty$;
(ii) if $\prod_{j=1}^\infty g_j : U \to \mathbb{C}$ is also locally uniformly converging, then so is $\prod_{j=1}^\infty f_j g_j$;
(iii) if each f_j is holomorphic then so is $\prod_{j=1}^\infty f_j$.

Proof. (i) By Lemma 3.2 the sequence $h_n(z) = \prod_{j=n}^\infty f_j(z)$ is pointwise well-defined. Fix $z_0 \in U$ and let $r > 0$ and $j_0 \in \mathbb{N}$ be as in Definition 3.6. The continuity of each f_j and the local uniform convergence imply that h_{j_0} is continuous. Since $h_{j_0}(z) \neq 0$ for all $z \in B_r(z_0)$ it holds that

$$\inf_{z \in B_{r/2}(z_0)} |h_{j_0}(z)| =: 2c > 0.$$

Since $\prod_{j=j_0}^{n-1} f_j(z) \to h_{j_0}$ uniformly on $B_r(z_0)$, there is an index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\inf_{z \in B_{r/2}(z_0)} \left| \prod_{j=j_0}^{n-1} f_j(z) \right| > c.$$

Then, for $n > \max\{j_0, n_0\}$ and $z \in B_{r/2}(z_0)$,

$$|h_n(z) - 1| = \left| \frac{h_{j_0}(z)}{\prod_{j=j_0}^{n-1} f_j(z)} - 1 \right| \leq \frac{1}{c} \left| h_{j_0}(z) - \prod_{j=j_0}^{n-1} f_j(z) \right|.$$

This proves the first part of (i), since by assumption the last term vanishes uniformly on $B_{r/2}(z_0)$ for $n \to \infty$. The second part of (i) follows from the first one since $f_j = \frac{h_j}{h_{j+1}}$.

(ii) This follows essentially from the definition, since the product of two locally uniformly converging sequences that are locally uniformly bounded still converges locally uniformly.

(iii) Apply Theorem 1.3. □

One drawback of local uniform convergence of products is that there is no invariance under rearrangement, i.e., the limit of infinite products might depend on the order of the sequence f_j. Hence we introduce the more stable notion of local normal convergence, relying on Lemma 3.5.

Definition 3.8. An infinite product of the form $\prod_{j=1}^{\infty}(1+g_j)$ with $g_j : U \to \mathbb{C}$ is called locally normally convergent if the series $\sum_{j=1}^{\infty} g_j$ is locally normally convergent (cf. Definition 1.10).

We next prove that local normal convergence of products implies local uniform convergence.

Lemma 3.9. *Assume that the product $\prod_{j=1}^{\infty}(1+g_j)$ converges locally normally. Then it also converges locally uniformly.*

Proof. Fix $z_0 \in U$ and let $r > 0$ be such that

$$\sum_{j=1}^{\infty} \sup_{z\in B_r(z_0)} |g_j(z)| < \infty.$$

Then there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have $\sup_{z\in B_r(z_0)} |g_j(z)| < \delta$, where $0 < \delta < \frac{1}{2}$ is chosen such that

$$\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z| \qquad \forall z \in B_\delta(0).$$

In particular, we have

$$\sum_{j\geq j_0} \sup_{z\in B_r(z_0)} |\log(1+g_j(z))| < \infty.$$

Hence by Lemma 1.11 the series $\sum_{j\geq j_0} \log(1+g_j)$ converges uniformly on $B_r(z_0)$. Taking the exponential yields the claim, as the exponential function never vanishes and is continuous. □

Remark 3.10. If $\prod_{j=1}^{\infty}(1+g_j)$ converges locally normally, then considering the series $\sum_{j=1}^{\infty} g_j$ we see that local normal convergence (of the series, hence also of the product) is invariant under rearrangements of the sequence $\{g_j\}_{j\in\mathbb{N}}$.

In Chapter 4 we will analyze the zeros of infinite products that converge locally normally. Given a holomorphic function $f : U \to \mathbb{C}$ we denote by $Z(f)$ the set of its zeros and by $o_c(f) \in \mathbb{N} \cup \{0, \infty\}$ the order of a zero $c \in U$ (with the convention that $o_c(f) = 0$ means $f(c) \neq 0$, while $o_c(f) = \infty$ if and only if f vanishes in a neighborhood of c).

Note that if $f_1, \dots, f_N : U \to \mathbb{C}$ is a finite family of such functions, then

$$Z(f_1 \cdot \ldots \cdot f_N) = \bigcup_{i=1}^{N} Z(f_j), \qquad o_c(f_1 \cdot \ldots \cdot f_N) = \sum_{i=1}^{N} o_c(f_i).$$

In the result below we generalize these identities to infinite products that converge locally uniformly.

Lemma 3.11. *Let $f_j : U \to \mathbb{C}$ be a sequence of holomorphic functions. Assume that $f = \prod_{j=1}^{\infty} f_j : U \to \mathbb{C}$ converges locally uniformly. Then*

$$Z(f) = \bigcup_{j=1}^{\infty} Z(f_j), \qquad o_c(f) = \sum_{j=1}^{\infty} o_c(f_j) \quad \forall c \in U.$$

Proof. Fix $c \in U$. Since $\prod_{j=1}^{\infty} f_j(c)$ converges, there exists $j_0 \in \mathbb{N}$ such that $f_j(c) \neq 0$ for all $j \geq j_0$. Write

$$f = f_1 \cdot \ldots \cdot f_{j_0-1} \cdot \underbrace{\prod_{j=j_0}^{\infty} f_j}_{=:g}.$$

Since g is holomorphic due to the local uniform convergence, and $g(c) \neq 0$, we conclude that

$$o_c(f) = \sum_{j=1}^{j_0-1} o_c(f_j) + o_c(g) = \sum_{j=1}^{\infty} o_c(f_j).$$

The previous equality also proves that $Z(f) = \bigcup_{j=1}^{\infty} Z(f_j)$.

□

In the exercise class we will show the product formula

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big).$$

For the proof we need the logarithmic derivative of infinite products, which will be the last topic of this chapter. Recall that the logarithmic derivative of a holomorphic function $f : U \to \mathbb{C}$ ($f \not\equiv 0$) is by definition the holomorphic function $h : U \setminus Z(f) \to \mathbb{C}$ given by $h = \frac{f'}{f}$. For infinite products the logarithmic derivative has a special structure.

Proposition 3.12. *Let $f_j : U \to \mathbb{C}$ be a sequence of holomorphic functions such that the product $f = \prod_{j=1}^{\infty} f_j : U \to \mathbb{C}$ converges locally normally. Then the logarithmic derivative $\frac{f'}{f} : U \setminus Z(f) \to \mathbb{C}$ is given by*

$$\frac{f'}{f} = \sum_{j=1}^{\infty} \frac{f_j'}{f_j},$$

where the series $\sum_{j=1}^{\infty} \frac{f_j'}{f_j}$ converges locally normally on $U \setminus Z(f)$.

Proof. Note that all functions $h_j = \frac{f_j'}{f_j}$ are holomorphic on $U \setminus Z(f)$. Moreover, we can write

$$f = f_1 \cdot \ldots \cdot f_{n-1} \cdot \underbrace{\prod_{j=n}^{\infty} f_j}_{=:g_n},$$

with g_n holomorphic due to local normal convergence and $g_n(z) \neq 0$ for all $z \in U \setminus Z(f)$. By iteration of the standard product rule we then calculate

$$\frac{f'}{f} = \frac{\sum_{j=1}^{n-1} \prod_{\substack{\ell=1 \\ \ell\neq j}}^{n-1} f_\ell \cdot f_j' g_n + \prod_{j=1}^{n-1} f_j \cdot g_n'}{\prod_{j=1}^{n-1} f_j \cdot g_n} = \sum_{j=1}^{n-1} \frac{f_j'}{f_j} + \frac{g_n'}{g_n}.$$

Combining Lemma 3.9 and Corollary 3.7 (i), we know that g_n converges locally uniformly to 1. From Theorem 1.5 we infer that also $g_n' \to 0$ locally uniformly. In particular, since g_n converges to a non-vanishing function, this implies that the logarithmic derivative $\frac{g_n'}{g_n}$ converges locally uniformly to 0. Thus

$$\frac{f'}{f} = \lim_{n\to\infty} \sum_{j=1}^{n-1} \frac{f_j'}{f_j} = \sum_{j=1}^{\infty} \frac{f_j'}{f_j}$$

and the series converges locally uniformly on $U \setminus Z(f)$.

It remains to show that $\sum_{j=1}^{\infty} \frac{f_j'}{f_j}$ converges locally normally on $U \setminus Z(f)$. Fix $z_0 \in U \setminus Z(f)$ and let $r > 0$ be such that $\overline{B_{2r}(z_0)} \subset U \setminus Z(f)$. By Corollary 3.7 (i) the sequence f_j converges locally uniformly to 1 on U. Hence we find $r > 0$ and an index $j_0 \in \mathbb{N}$ such that $|f_j(z)| \geq \frac{1}{2}$ for all $j \geq j_0$ and $z \in B_r(z_0)$. Setting $h_j = f_j - 1$ we conclude that

$$\sum_{j=j_0}^{\infty} \sup_{z\in B_r(z_0)} \left|\frac{f_j'(z)}{f_j(z)}\right| \leq 2 \sum_{j=j_0}^{\infty} \sup_{z\in B_r(z_0)} |h_j'(z)| \leq \frac{4}{r} \sum_{j=j_0}^{\infty} \sup_{w\in\partial B_{2r}(z_0)} |h_j(w)|, \tag{6}$$

where in the last inequality we used the standard Cauchy estimate for derivatives, obtained from Corollary 0.3 in the form

$$|h_j'(z)| = \left|\frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{h_j(w)}{(w-z)^2}\, dw\right| \leq \frac{2}{r} \sup_{w\in\partial B_{2r}(z_0)} |h_j(w)| \qquad \forall z \in B_r(z_0).$$

By the local normal convergence of the infinite product, if $r > 0$ is chosen sufficiently small then the last sum in (6) is finite. This proves the claim.

$\square$

Remark 3.13.

(i) Proposition 3.12 holds verbatim if we replace local normal convergence by local uniform convergence everywhere. The proof remains unchanged except that we do not need the last argument.

(ii) Even if each $f_j \not\equiv 0$, it can happen that $Z(f) = U$ when U is not connected (take for each connected component an f_j that vanishes only on that component). If instead U is connected, then $Z(f_j)$ has no accumulation points in U, hence is at most a countable set (check this!), so that $Z(f)$ can be at most countable.

This was the last basic result on infinite products that we need. Next we apply them to prove the celebrated product theorem of Weierstrass.

4. The Weierstrass product theorem

The set of zeros of a non-constant entire function is always closed and discrete, by the identity theorem. In this chapter we study the reverse problem: given a closed discrete set $S = \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $Z(f) = S$, with prescribed multiplicity at each zero? If the set S is finite, say $a_1, \dots, a_N$ (with multiple occurrences allowed), then the polynomial

$$P(z) = \prod_{j=1}^{N} (z - a_n)$$

satisfies all properties. The Weierstrass product theorem gives an existence result in the infinite case.

In general one cannot expect the convergence of the infinite product

$$\prod_{n=1}^{\infty} \Big(1 - \frac{z}{a_n}\Big).$$

Hence we add factors $g_n : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ such that $(1 - \frac{z}{a_n}) g_n(\frac{z}{a_n})$ is close to 1. Since $|a_n| \to \infty$, it suffices to note that for $|z| < 1$ we have

$$(1 - z)e^{-\log(1-z)} = 1.$$

Hence we consider a suitable Taylor polynomial of the function $z \mapsto -\log(1 - z)$ at the origin. Note that for $|z| \leq \frac{1}{2}$ we can write

$$-\log(1 - z) = \sum_{k=1}^{\infty} \frac{z^k}{k}.$$

This motivates the definition of the so-called Weierstrass factors given by

$$E_0(z) = 1 - z \qquad \text{and} \qquad E_n(z) = (1 - z) \cdot e^{\sum_{k=1}^{n} \frac{z^k}{k}} \quad \text{for } n \in \mathbb{N}.$$

The following estimate turns out to be very useful for proving the Weierstrass product theorem.

Lemma 4.1. *We have $|E_n(z) - 1| \leq |z|^{n+1}$ for all $z \in B_1(0)$.*

Proof. The claim is trivial for $n = 0$, hence fix $n \geq 1$. To reduce notation we set $p_n(z) = \sum_{k=1}^{n} \frac{z^k}{k}$. Then

$$(1 - z)p_n'(z) = (1 - z)\sum_{k=0}^{n-1} z^k = 1 - z^n$$

and therefore by the product rule

$$E_n'(z) = -e^{p_n(z)} + (1 - z)p_n'(z)e^{p_n(z)} = -z^n e^{p_n(z)}.$$

On the other hand, denoting by $\sum_{k=0}^{\infty} a_k z^k$ the Taylor series of E_n at the origin (which converges in all of $\mathbb{C}$) we have that

$$\sum_{k=0}^{\infty} k a_k z^{k-1} = E_n'(z) = -z^n e^{p_n(z)}.$$

The right hand side term has a zero of order n at $z=0$. Hence we conclude that

$$a_k = 0 \qquad \forall 1 \le k \le n.$$

Moreover, as the coefficients of the Taylor series of $z \mapsto e^{p_n(z)}$ are all non-negative, we conclude that

$$|a_k| = -a_k \qquad \forall k > n.$$

Since $1 = E_n(0) = a_0$ and therefore $0 = E_n(1) = 1 + \sum_{k>n} a_k$, we conclude by Hölder's inequality that if $|z|<1$ then

$$|E_n(z) - 1| \le \sum_{k=n+1}^{\infty} |a_k||z|^k \le \underbrace{\sup_{k>n} |z|^k}_{=|z|^{n+1}} \cdot \sum_{k=n+1}^{\infty} |a_k| = -|z|^{n+1} \cdot \underbrace{\sum_{k=n+1}^{\infty} a_k}_{=-1} = |z|^{n+1}.$$

This concludes the proof.

□

With the previous lemma at hand we can now prove the Weierstrass product theorem on $\mathbb{C}$. Note that the existence result remains valid on any open set, but the structure of the function f will be slightly different (cf. Exercise H 7.3).

Theorem 4.2 (Weierstrass product theorem). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers with no accumulation points in $\mathbb{C}$. For each $n \in \mathbb{N}$ set $o_n := \#\{k \in \mathbb{N} : a_k = a_n\}$. Assume that $a_n \neq 0$ and $o_n < \infty$ for all $n \in \mathbb{N}$. Then*

$$f(z) := \prod_{n=1}^{\infty} E_n(\tfrac{z}{a_n}) = \prod_{n=1}^{\infty} \Big(1 - \frac{z}{a_n}\Big) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n}$$

converges locally normally and defines an entire function with $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbb{N}$.

Remark 4.3. Note that the function $z \mapsto z^k f(z)$ allows us to add a zero at $z=0$ with arbitrary multiplicity $k \in \mathbb{N}$ to the above result.

Proof of Theorem 4.2. We show that the infinite product defining f converges locally normally on $\mathbb{C}$. To this end, fix a compact set $K \subset \mathbb{C}$. First note that $\lim_{n\to\infty} |a_n| = \infty$. Hence there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{z}{a_n}\right| \le \frac{1}{2}$ for all $z \in K$ and $n \ge n_0$. In particular, we can apply Lemma 4.1 to deduce that

$$\sum_{n=n_0}^{\infty} \sup_{z\in K} |E_n(\tfrac{z}{a_n}) - 1| \le \sum_{n=n_0}^{\infty} 2^{-(n+1)} < \infty.$$

By definition this shows the local normal convergence of the product. Corollary 3.7(iii) then implies that the function f is entire. Moreover, by Lemma 3.11 we know that

$$Z(f) = \bigcup_{n=1}^{\infty} Z\big(E_n(\tfrac{\cdot}{a_n})\big) = \{a_n\}_{n\in\mathbb{N}}, \qquad o_{a_n}(f) = \sum_{j=1}^{\infty} o_{a_n}\big(E_n(\tfrac{\cdot}{a_n})\big) = o_n,$$

where we used that each E_n has a simple zero at $z=1$ and no other zeros.

□

The Weierstrass product theorem implies the following representation result of entire functions.

Corollary 4.4. *Let $f \not\equiv 0$ be an entire function and write its zeros in $\mathbb{C} \setminus \{0\}$ as the multiset $S := \{a_1, a_2, a_3, \ldots\}$. Then*

$$f(z) = e^{h(z)} z^m \prod_{n=1}^{|S|} E_n\left(\tfrac{z}{a_n}\right),$$

where h is an entire function, $m = o_0(f)$, and the product converges locally normally in $\mathbb{C}$.

Proof. Applying the Weierstrass product theorem (or its finite analogue) to the sequence $\{a_n\}_n$ yields that the function

$$g(z) = z^{o_0(f)} \prod_{n=1}^{|S|} E_n\left(\tfrac{z}{a_n}\right)$$

is entire with $Z(g) = Z(f)$ and $o_z(g) = o_z(f)$ for all $z \in \mathbb{C}$. Hence the quotient f/g has only removable singularities and therefore represents an entire function that never vanishes. It is a well-known result from complex analysis that on the simply connected domain $\mathbb{C}$ this implies that $f/g = e^h$ for some entire function h (see Corollary 6.10). This finishes the proof. ∎

5. The Hadamard factorization theorem

In this section we show that if an entire function does not grow too fast, then the product representation obtained through the Weierstrass theorem in Corollary 4.4 can be significantly refined, in the sense that it will suffice to take Weierstrass factors $E_k(\frac{z}{a_n})$ with a fixed index k. We closely follow [5, Chapter 5].

For the proof of convergence of such products we will need to relate the size of the zeros of f to the growth of f itself. The bridge between the two concepts if provided by Jensen's formula.

Theorem 5.1 (Jensen's formula)**.** *Let $R > 0$ and U be an open set containing $\overline{B_R(0)}$. Suppose that $f : U \to \mathbb{C}$ is holomorphic, $f(0) \neq 0$, and $f(z) \neq 0$ for $|z| = R$. Denote by $z_1, z_2, \ldots, z_N$ the zeros (with multiplicity) of f in $B_R(0)$. Then*

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta.$$

Proof. First obeserve that if f_1 and f_2 satisfy the theorem, then $f_1 f_2$ does as well. Indeed it is clear that $f_1 f_2$ satisfies the hypotheses, and the conclusion follows from $\log(xy) = \log(x) + \log(y)$ for $x, y \in \mathbb{R}_{>0}$.

Now consider the function

$$g(z) = \frac{f(z)}{(z - z_1)\cdots(z - z_N)}.$$

Then g is holomorphic in U, since it has removable singularities at the z_k. Furthermore it is non-vanishing in $\overline{B_R(0)}$, since all of the zeros of f there are canceled by the factors $z - z_j$. By the previous observation, it suffices to prove Jensen's formula for $g(z)$ and for $z - z_j$.

Let us prove the formula for $g(z)$, which becomes

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\, d\theta. \tag{7}$$

Observe that g is non-vanishing in a slightly larger open ball $B_{R+\varepsilon}(0) \subset U$, so in that ball we have $g(z) = e^{h(z)}$ for some holomorphic $h : B_{R+\varepsilon}(0) \to \mathbb{C}$, by Corollary 6.10. Then

$$|g(z)| = |e^{h(z)}| = |e^{\mathrm{Re}(h(z)) + \mathrm{Im}(h(z))}| = e^{\mathrm{Re}(z)}, \qquad \text{so} \qquad \log|g(z)| = \mathrm{Re}(h(z)). \tag{8}$$

By Cauchy's formula and the change of variables $z \mapsto Re^{i\theta}$, we have have the mean value identity

$$h(0) = \frac{1}{2\pi i}\int_{\partial B_R(0)} \frac{h(z)}{z}\, dz = \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\, d\theta, \tag{9}$$

and (7) follows after taking real parts in (9) and using (8).

Finally, we are left with proving the theorem for $f(z) = z - w$, with $|w| < R$. The desired conclusion is

$$\log|w| = \log\left(\frac{|w|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|\, d\theta.$$

From $\log(|w|/R) = \log|w| - \log R$ and $\log|Re^{i\theta} - w| = \log R + \log|e^{i\theta} + w/R|$, it suffices to show that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - a|\, d\theta = 0, \qquad \text{whenever } |a| < 1.$$

Changing variables $\theta \mapsto -\theta$, this is equivalent to

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1 - ae^{i\theta}|\, d\theta = 0, \qquad \text{whenever } |a| < 1.$$

To prove this, we argue as before but now for the function $F(z) = 1 - az$. Since F is non-vanishing in $\overline{B_1(0)}$, there exists G holomorphic in an open ball of radius greater than 1 such that $F(z) = e^{G(z)}$. Then $\log|F(z)| = \operatorname{Re}(G(z))$, and $\log|F(0)| = \log 1 = 0$. Applying the mean value identity to $G(z)$, coming from Cauchy's formula as in (9) but on $\partial B_a(0)$, concludes the proof of the theorem.

□

Now we derive consequences for the number of zeros of f in balls around the origin. If $f \not\equiv 0$ is holomorphic in $B_R(0)$, then for each $0 \le r < R$ we denote by

$$N_f(r) := \sum_{z \in Z(f)\cap B_r(0)} o_z(f)$$

the number of zeros of f (with multiplicity) whose absolute value is less than r. Observe that $N_f(r)$ is a non-decreasing function of r.

Corollary 5.2. *Let $R > 0$ and U be an open set containing $\overline{B_R(0)}$. Suppose that $f : U \to \mathbb{C}$ is holomorphic, $f(0) \neq 0$, and $f(z) \neq 0$ for $|z| = R$. Then*

$$\int_0^R N_f(x)\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)|.$$

Proof. Let $z_1, z_2, \ldots, z_N$ be the zeros of f inside $B_R(0)$, and consider the indicator functions $\eta_k(x) := \mathbf{1}(x > |z_k|)$. Then

$$\int_0^R N_f(x)\frac{dx}{x} = \int_0^R \left(\sum_{k=1}^N \eta_k(x)\right)\frac{dx}{x} = \sum_{k=1}^N \int_{|z_k|}^R \frac{dx}{x} = \sum_{k=1}^N \log\left|\frac{R}{z_k}\right|.$$

The result follows directly from Jensen's formula and the identity above.

□

Definition 5.3. Let f be an entire function. If there exist $\rho > 0$ and constants $A, B > 0$ (possibly depending on f and ρ) such that

$$|f(z)| \le Ae^{B|z|^\rho} \qquad \text{for all } z \in \mathbb{C},$$

then we say that f has an order of growth $\le \rho$. In that case, we define *the* order of growth of f as $\rho_f = \inf \rho$, where the infimum is taken over all $\rho > 0$ such that f has an order of growth $\le \rho$.

Remark 5.4. If the order of growth of f is ρ_f, then f need not have an order of growth $\le \rho_f$ (as the infimum is not always attained).

Lemma 5.5. *If $f \not\equiv 0$ is an entire function that has an order of growth $\le \rho$, then*

(i) $N_f(r) \le Cr^\rho$ for some $C > 0$ and all $r \ge 1$;

(ii) If $S := \{a_1, a_2, a_3, \ldots\}$ denotes the multiset of zeros of f (including multiplicities) with $a_n \neq 0$, then for any $s > \rho$ we have

$$\sum_{n=1}^{|S|} \frac{1}{|a_n|^s} < \infty.$$

Proof. Let us start with (i). We may assume that $f(0) \neq 0$, since denoting $\ell = o_0(f)$ and $F(z) = f(z)/z^\ell$, $N_F(r)$ and $N_f(r)$ differ only by ℓ (a constant) and F also has an order of growth $\leq \rho$.

If $f(0) \neq 0$, then applying Corollary 5.2 with $R = 2r$ gives

$$\int_r^{2r} N_f(x)\frac{dx}{x} \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\theta})|\, d\theta - \log|f(0)|.$$

But since $N_f(x)$ is non-decreasing we have

$$\int_r^{2r} N_f(x)\frac{dx}{x} \geq N_f(r)\int_r^{2r}\frac{dx}{x} = N_f(r)\log 2.$$

From the growth condition on f we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\theta})|\, d\theta \leq \log\left(Ae^{B(2r)^\rho}\right) = (2^\rho B)r^\rho + \log A.$$

Thus we obtain $N_f(r) \leq Cr^\rho$ for a constant $C > 0$ (depending on f and ρ) and all $r \geq 1$.

To prove (ii), we use (i) to bound

$$\begin{aligned}\sum_{|a_n|\geq 1} |a_n|^{-s} &= \sum_{j=0}^{\infty}\left(\sum_{2^j\leq|a_n|<2^{j+1}} |a_n|^{-s}\right) \leq \sum_{j=0}^{\infty} N_f(2^{j+1})2^{-js}\\ &\leq C\sum_{j=0}^{\infty}(2^{j+1})^\rho 2^{-js} = C2^\rho\sum_{j=0}^{\infty}(2^{\rho-s})^j < \infty,\end{aligned}$$

since $s > \rho$.

□

Remark 5.6. The condition $s > \rho$ in Lemma 5.5 cannot be improved. For instance, $f(z) = \sin(\pi z)$ has an order of growth ≤ 1 and zeros at $z = n \in \mathbb{Z}$, but $\sum_{n\neq 0}\frac{1}{|n|^s}$ converges precisely when $s > 1$.

We are now ready to state Hadamard's theorem.

Theorem 5.7 (Hadamard factorization theorem)**.** *Let $f \not\equiv 0$ be an entire function with finite order of growth ρ_f, and denote $k = \lfloor \rho_f \rfloor$. Write the zeros of f in $\mathbb{C}\setminus\{0\}$ as the multiset $S := \{a_1, a_2, a_3, \ldots\}$. Then*

$$f(z) = e^{P(z)}z^m\prod_{n=1}^{|S|} E_k\left(\frac{z}{a_n}\right),$$

where P is a polynomial of degree $\leq k$, $m = o_0(f)$, and the product converges locally normally in $\mathbb{C}$.

The proof of local normal convergence of the product is not hard using the tools we have developed so far. However, to show that P is a polynomial we will need upper bounds for the corresponding factor $e^{P(z)}$, which will require lower bounds for the Weierstrass factors $E_k(z/a_n)$.

Lemma 5.8. *For each integer $k \geq 0$ there is a constant $C_k > 0$ such that*

$$|E_k(z)| \geq e^{-C_k|z|^{k+1}} \qquad \text{if } |z| \leq \frac{1}{2}$$

and

$$|E_k(z)| \geq |1-z|\cdot e^{-C_k|z|^k} \qquad \text{if } |z| \geq \frac{1}{2}.$$

Proof. If $|z| \leq \frac{1}{2}$ then Lemma 4.1 gives $|E_k(z) - 1| \leq |z|^{k+1}$, so that $|E_k(z)| \geq 1 - |z|^{k+1}$. Then the first inequality (with $C_k = 2$) follows from $1 - x \geq e^{-2x}$ for $0 \leq x \leq \frac{1}{2}$, since in that range we have

$$e^{2x} = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} \geq 1 + 2x \geq \frac{1}{1-x}.$$

If $|z| \geq \frac{1}{2}$ then observing that

$$|E_k(z)| = |1-z| \cdot e^{\mathrm{Re}\left(\sum_{n=1}^k z^n/n\right)} \geq |1-z| \cdot e^{-\left|\sum_{n=1}^k z^n/n\right|},$$

we conclude that the second inequality (for instance with $C_k = 2^k$) follows from

$$\Big|\sum_{n=1}^{k} \frac{z^n}{n}\Big| \leq \sum_{n=1}^{k} |z|^n \leq C_k |z|^k.$$

□

Using the notation of Theorem 5.7, we obtain the following lower bound for the Weierstrass product away from the zeros.

Lemma 5.9. *For any $s \in \mathbb{R}$ with $\rho_f < s < k+1$, there exists $C > 0$ such that*

$$\left|\prod_{n=1}^{|S|} E_k\Big(\frac{z}{a_n}\Big)\right| \geq e^{-C|z|^s}$$

for all $z \in \mathbb{C}$ satisfying $|z| \geq 1$ and $|z - a_n| \geq |a_n|^{-k-1}$ for all $n \in \mathbb{N}$.

Proof. Split the product as

$$\prod_{n=1}^{|S|} E_k\Big(\frac{z}{a_n}\Big) = \prod_{|a_n| \leq 2|z|} E_k\Big(\frac{z}{a_n}\Big) \prod_{|a_n| > 2|z|} E_k\Big(\frac{z}{a_n}\Big).$$

The second product can be directly bounded using Lemma 5.8, which implies

$$\left|\prod_{|a_n|>2|z|} E_k\Big(\frac{z}{a_n}\Big)\right| = \prod_{|a_n|>2|z|} \left|E_k\Big(\frac{z}{a_n}\Big)\right| \geq \prod_{|a_n|>2|z|} e^{-C_k|z/a_n|^{k+1}} = e^{-C_k|z|^{k+1}\sum_{|a_n|>2|z|}|a_n|^{-k-1}}.$$

Since $|a_n| > 2|z|$ and $s < k+1$, we have

$$|a_n|^{-k-1} = |a_n|^{-s}|a_n|^{s-k-1} \leq C'|a_n|^{-s}|z|^{s-k-1}.$$

But $s > \rho_f$, so Lemma 5.5 (ii) implies convergence of $\sum_n |a_n|^{-s}$ and we obtain the desired bound

$$\left|\prod_{|a_n|>2|z|} E_k\Big(\frac{z}{a_n}\Big)\right| \geq e^{-C|z|^s}$$

for some $C > 0$.

It remains to estimate the first product. From Lemma 5.8 we obtain

$$\left|\prod_{|a_n|\leq 2|z|} E_k\Big(\frac{z}{a_n}\Big)\right| \geq \prod_{|a_n|\leq 2|z|} \left|1 - \frac{z}{a_n}\right| \prod_{|a_n|\leq 2|z|} e^{-C_k|z/a_n|^k}. \tag{10}$$

Since $|a_n|^{-k} = |a_n|^{-s}|a_n|^{s-k} \leq C'|a_n|^{-s}|z|^{s-k}$, we conclude as before that

$$\prod_{|a_n|\leq 2|z|} e^{-C_k|z/a_n|^k} = e^{-C_k|z|^k\sum_{|a_n|\leq 2|z|}|a_n|^{-k}} \geq e^{-C|z|^s}$$

for some $C > 0$. Finally, for the first term of (10) we will need the assumption that z is away from the zeros. Indeed, from the condition $|z - a_n| \geq |a_n|^{-k-1}$ for all $n \in \mathbb{N}$ we obtain

$$\prod_{|a_n|\leq 2|z|} \left|1 - \frac{z}{a_n}\right| = \prod_{|a_n|\leq 2|z|} \left|\frac{a_n - z}{a_n}\right| \geq \prod_{|a_n|\leq 2|z|} |a_n|^{-k-2}.$$

To estimate the remaining product we observe from Lemma 5.5 (i) that since $s > \rho_f$ and $|z| \geq 1$ we have

$$(k+2)\sum_{|a_n|\leq 2|z|} \log|a_n| \leq (k+2)N_f(2|z|)\log(2|z|) \leq C|z|^s$$

for some $C > 0$. This finishes the proof of the lemma.

□

Corollary 5.10. *There is a sequence $\{r_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ of radii with $r_m \to \infty$ as $m \to \infty$ such that for every $m \in \mathbb{N}$ we have*

$$\left| \prod_{n=1}^{|S|} E_k\Big(\frac{z}{a_n}\Big) \right| \geq e^{-C|z|^s} \qquad \text{for } |z| = r_m.$$

Proof. Let $\mathcal{R} \subset \mathbb{R}_{>0}$ denote the set of radii $r > 0$ such that the circle of radius r around the origin intersects some open ball $B_{|a_n|^{-k-1}}(a_n)$ prohibited by Lemma 5.9, i.e. such that

$$|a_n| - |a_n|^{-k-1} < r < |a_n| + |a_n|^{-k-1}$$

for some n. Since $\sum_{n=1}^{|S|} |a_n|^{-k-1} < \infty$ by Lemma 5.5 (ii), the set $\mathcal{R}$ (which is a countable union of open intervals) has finite measure. Thus for every sufficiently large $m \in \mathbb{N}$ there exist a radius $r_m \in [m, m+1)$ such that $r_m \notin \mathcal{R}$. Therefore Lemma 5.9 holds if $|z| = r_m$ and gives the desired bound.

□

We are ready to prove Hadamard's theorem.

Proof of Theorem 5.7. Let

$$g(z) = z^m \prod_{n=1}^{|S|} E_k\Big(\frac{z}{a_n}\Big).$$

The local normal convergence of the product above follows as in the proof of Theorem 4.2, since for any compact $K \subset \mathbb{C}$ we have by Lemma 4.1 that

$$\sup_{z\in K} \left|1 - E_k\Big(\frac{z}{a_n}\Big)\right| \leq \sup_{z\in K} \left|\frac{z}{a_n}\right|^{k+1}$$

for all n sufficiently large in terms of K (so that $|a_n| > \sup_{z\in K} |z|$). Then the product defining g converges locally normally since $\sum_{n=1}^{|S|} |a_n|^{-k-1} < \infty$.

Also note that f/g is entire and non-vanishing, as the zeros of f exactly match those of g, so $f(z) = g(z)e^{h(z)}$ for some entire function h. Since the order of growth of f is $\rho_f < s$ and we have the lower bound of Corollary 5.10, observe that there are $A, B > 0$ such that whenever $|z| = r_m$ for some m then

$$e^{\mathrm{Re}(h(z))} = \left|\frac{f(z)}{g(z)}\right| \leq Ae^{B|z|^s}.$$

This implies

$$\mathrm{Re}(h(z)) \leq C|z|^s \qquad \text{whenever } |z| = r_m, \tag{11}$$

for some $C > 0$ and every sufficiently large m. The proof of the theorem will be finished once we show that h must be a polynomial of degree $\leq s$ (as $s < k + 1$).

Since h is entire it has a series expansion

$$h(z) = \sum_{n=0}^{\infty} c_n z^n$$

for $z \in \mathbb{C}$. Then for any $n \in \mathbb{Z}$ and $r > 0$, Cauchy's formula gives

$$c_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{h(z)}{z^{n+1}}\, dz = \frac{1}{2\pi r^n} \int_0^{2\pi} h(re^{i\theta})e^{-in\theta}\, d\theta.$$

Now assume $n \geq 1$. Since $c_{-n} = 0$ and $2\,\mathrm{Re}(h) = h + \overline{h}$, we have

$$c_n r^n = c_n r^n + \overline{c_{-n} r^{-n}} = \frac{1}{\pi} \int_0^{2\pi} \mathrm{Re}(h(re^{i\theta}))e^{-in\theta}\, d\theta.$$

Applying this with $r = r_m$ for m sufficiently large, the bound (11) gives

$$|c_n| \leq 2C(r_m)^{s-n}.$$

Since $r_m \to \infty$ as $m \to \infty$, taking the limit yields $c_n = 0$ for every $n > s$. Therefore h is a polynomial of degree $\leq s$, as desired.

□

6. Picard's little and great theorems

We now come to two celebrated theorems of complex analysis in one variable. They provide a very fine description of the image of entire functions (Picard's little theorem) or the image of neighborhoods of an essential singularity (Picard's great theorem). We will see that the little theorem follows (in an even stronger form) from the great theorem. Let us first formulate them.

Theorem 6.1 (Picard's little theorem)**.** *Let f be a non-constant entire function. Then f assumes each value in $\mathbb{C}$ except at most one.*

Theorem 6.2 (Picard's great theorem)**.** *Let $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and have an essential singularity at $z_0 \in \mathbb{C}$, for $r > 0$. Then in the punctured neighborhood $B_r(z_0) \setminus \{z_0\}$ the function f assumes every value in $\mathbb{C}$, except at most one, infinitely many times.*

Proof of Theorem 6.1 assuming Theorem 6.2. If f is a polynomial then the claim follows by the fundamental theorem of algebra, as $z \mapsto f(z) - w$ has a zero for every $w \in \mathbb{C}$, hence f is surjective. If f is not a polynomial, then $z \mapsto f(\frac{1}{z})$ has an essential singularity at $z = 0$ (this follows from the classification of isolated singularities via the principal part of the Laurent series expansion, in Corollary 0.11). Hence the claim follows from Theorem 6.2, as the map $z \mapsto \frac{1}{z}$ is one-to-one from $\mathbb{C} \setminus \{0\}$ onto itself.

□

Remark 6.3.

(i) The proof above shows that if f is not a polynomial then in fact it assumes each value of $\mathbb{C}$, except at most one, infinitely many times.
(ii) The function $f(z) = e^z$ shows that Theorem 6.1 is sharp, since it avoids 0.

Next we turn to the proof of Picard's great theorem. On the way we prove several results that are interesting on their own. Picard's great theorem will then be a consequence of a strengthened version of Montel's compactness theorem. Let us mention that Picard proved the two theorems by different means.

We start with Bloch's theorem, which provides large balls contained in the image of non-constant holomorphic functions. In what follows $\mathcal{H}(\overline{U})$ denotes the set of functions which are holomorphic in some neighborhood of $\overline{U}$.

Theorem 6.4 (Bloch's theorem)**.** *Let $f \in \mathcal{H}(\overline{B_1(0)})$ be such that $f'(0) = 1$. Then there exists $p \in \mathbb{C}$ such that $B_{\frac{3}{2}-\sqrt{2}}(p) \subset f(B_1(0))$.*

Remark 6.5. There is no universal $\delta > 0$ such that $B_\delta(0) \subset f(B_1(0))$ for all f as in Theorem 6.4. Indeed, the functions $f_\varepsilon(z) = \varepsilon(e^{z/\varepsilon} - 1)$ satisfy $f_\varepsilon'(0) = 1$ for all $\varepsilon > 0$, but $-\varepsilon \notin f_\varepsilon(B_1(0))$.

Proof of Theorem 6.4. We divide the proof into three steps. The first two are more general statements.

Step 1: We show that if $D \subset \mathbb{C}$ is a bounded domain, $g \in \mathcal{H}(\overline{D})$ is non-constant, and $a \in D$ is such that $s := \inf_{z \in \partial D} |g(z) - g(a)| > 0$, then $B_s(g(a)) \subset g(D)$.

Indeed, due to the boundedness of g on D (since $\overline{D}$ is compact), the set $\partial g(D)$ is compact. Hence there exists $w \in \partial g(D)$ such that $\operatorname{dist}(\partial g(D), g(a)) = |w - g(a)|$. We argue that $|w - g(a)| \geq s$, which proves the first step. To this end, note that there exists a sequence $z_n \in D$ such that $g(z_n) \to w$, and without loss of generality we can also assume that $z_n \to z \in \overline{D}$ after passing to a convergent subsequence. Then by continuity $g(z) = w \in \partial g(D)$. The open mapping theorem (Theorem 0.7) implies $z \in \partial D$. Hence by definition $|w - g(a)| \geq s$.

Step 2: We show that if $g \in \mathcal{H}(\overline{B_r(a)})$ is non-constant and $\sup_{w \in B_r(a)} |g'(w)| \leq 2|g'(a)|$, then $B_R(g(a)) \subset g(B_r(a))$ for $R = (3 - 2\sqrt{2})r|g'(a)|$.

To prove this, first notice that upon considering the function $z \mapsto g(z+a) - g(a)$ we can assume that $a = g(a) = 0$. Then the function $A(z) = g(z) - g'(0)z$ satisfies

$$A(z) = \int_{[0,z]} g'(\zeta) - g'(0)\, d\zeta,$$

so that by the definition of the path integral (parametrizing $\zeta = tz$) we have the bound

$$|A(z)| \leq \int_0^1 |g'(tz) - g'(0)| \cdot |z|\, dt.$$

In order to bound the difference in the integrand we express it by Cauchy's integral formula as

$$g'(v) - g'(0) = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{g'(\zeta)}{\zeta - v} - \frac{g'(\zeta)}{\zeta}\, d\zeta = \frac{v}{2\pi i} \int_{\partial B_r(0)} \frac{g'(\zeta)}{\zeta(\zeta - v)}\, d\zeta \qquad \forall v \in B_r(0),$$

so that still for $v \in B_r(0)$ we have

$$|g'(v) - g'(0)| \leq \frac{|v|}{r - |v|} \sup_{\zeta \in \partial B_r(0)} |g'(\zeta)| \leq \frac{|v|}{r - |v|} \sup_{w \in B_r(0)} |g'(w)|.$$

Combined with our assumption $\sup_{w \in B_r(0)} |g'(w)| \leq 2|g'(0)|$, for $z \in B_r(0)$ this yields the bound

$$|A(z)| \leq \int_0^1 \frac{|tz|}{r - |tz|} \sup_{w \in B_r(0)} |g'(w)| \cdot |z|\, dt \leq \frac{1}{2} \frac{|z|^2}{r - |z|} \sup_{w \in B_r(0)} |g'(w)| \leq \frac{|z|^2}{r - |z|} |g'(0)|.$$

Here we used that $\eta(t) := \frac{t|z|}{r - t|z|}$ is convex on $[0,1]$, so that $\int_0^1 \eta(t)\, dt \leq \frac{1}{2}(\eta(0) + \eta(1)) = \frac{1}{2} \frac{|z|}{r - |z|}$. Since also $|A(z)| \geq |g'(0)||z| - |g(z)|$, we deduce that for all $z \in B_r(0)$ it holds that

$$|g(z)| \geq \left(|z| - \frac{|z|^2}{r - |z|}\right) |g'(0)|.$$

In order to apply Step 1 in the most efficient way we choose $|z| = \rho^*$ for which the term in brackets is maximal. With elementary analysis one can show that the real-valued function $\rho \mapsto \rho - \frac{\rho^2}{r - \rho}$ takes its maximum on $(0, r)$ at $\rho^* = (1 - \frac{1}{2}\sqrt{2})r$ with value $(3 - 2\sqrt{2})r$. Hence applying Step 1 with $D = B_{\rho^*}(0)$ and $a = g(a) = 0$ yields $B_R(0) \subset g(B_r(0))$ with $R = (3 - 2\sqrt{2})r|g'(0)|$, as claimed.

Step 3: Conclusion.

To the function $f \in \mathcal{H}(\overline{B_1(0)})$ we associate the function $h(z) := |f'(z)|(1 - |z|)$, which is continuous on $\overline{B_1(0)}$. Since by assumption $f'(0) = 1$, it follows that the maximum of h on $\overline{B_1(0)}$ is attained at some point $p \in B_1(0)$ with $M := h(p) \geq |f'(0)| = 1$. Setting $r = \frac{1}{2}(1 - |p|)$ we have $M = 2r|f'(p)|$ and $B_r(p) \subset B_1(0)$. Moreover, note that for $z \in B_r(p)$ it holds that

$$|z| \leq |p| + r = 1 - r,$$

or equivalently $1 - |z| \geq r$. Using the maximality $|f'(z)|(1 - |z|) \leq 2r|f'(p)|$ for $z \in \overline{B_1(0)}$, we conclude that $|f'(z)| \leq 2|f'(p)|$ for all $z \in B_r(p)$, so Step 2 implies that $B_R(f(p)) \subset f(B_1(0))$ for

$$R = (3 - 2\sqrt{2})r|f'(p)| = (\tfrac{3}{2} - \sqrt{2})M \geq (\tfrac{3}{2} - \sqrt{2}),$$

as claimed.

□

Remark 6.6. Computing the integral in Step 2 exactly, the proof above allows us to increase the constant $\frac{3}{2} - \sqrt{2} = 0.0857...$ in Bloch's theorem to $\frac{1}{2} - \log\left(\frac{3}{2}\right) = 0.0945....$ The largest possible constant is not known, but see Exercise H 9.3 for improvements and the best known bounds.

Bloch's theorem might seem quite restrictive when formulated only on the unit disc, but there are some straightforward consequences.

Corollary 6.7. *If $f : U \to \mathbb{C}$ is holomorphic and $f'(c) \neq 0$ at a point $c \in U$, then $f(U)$ contains a ball of radius $(\frac{3}{2} - \sqrt{2})s|f'(c)|$, for every $0 < s < \operatorname{dist}(c, \partial U)$. In particular, if $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then $f(\mathbb{C})$ contains balls of arbitrarily large radius.*

Proof. See Exercise H 9.1. □

Both theorems of Picard rule out functions that omit two (or more) values, so we need to study such functions in more detail. Before that, we recall the notion of a simply connected (open) set.

Definition 6.8. Let $G \subset \mathbb{C}$ be an open set. We say that G is simply connected if it is path-connected and every closed curve $\gamma \subset G$ can be contracted in G to a point. More precisely, for every continuous $\gamma : [0,1] \to G$ with $\gamma(0) = \gamma(1)$ there exists a point $z_0 \in G$ and a continuous map $H : [0,1] \times [0,1] \to G$ such that

(i) $H(0,t) = \gamma(t) \qquad \forall t \in [0,1]$;
(ii) $H(1,t) = z_0 \qquad \forall t \in [0,1]$;
(iii) $H(s,0) = H(s,1) \qquad \forall s \in [0,1]$.

We will rely on the fact that on simply connected domains, every holomorphic function has a primitive. The proof is given in any basic complex analysis course (e.g. [5, Chapter 3, Theorem 5.2]), so we omit it.

Theorem 6.9. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic. Then there exists a holomorphic function $F : G \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in G$.*

Based on this theorem we can show that on simply connected domains there always exist holomorphic logarithms and n-th roots.

Corollary 6.10. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C} \setminus \{0\}$ be holomorphic. Then there exists a holomorphic function $\log(f) : G \to \mathbb{C}$ such that $\exp(\log(f)) = f$. Moreover, for each $n \in \mathbb{N}$ there exists a holomorphic function $\sqrt[n]{f} : G \to \mathbb{C}$ such that $(\sqrt[n]{f})^n = f$.*

Proof. Consider the logarithmic derivative $h : G \to \mathbb{C}$, defined by $h = \frac{f'}{f}$, which is holomorphic on G since f does not vanish. Choose a primitive $H : G \to \mathbb{C}$ of h (i.e. H is holomorphic and $H' = h$) such that for some $z_0 \in G$ we have $H(z_0) = \log(f(z_0))$, where $\log(\cdot)$ denotes the principal branch of the logarithm. This can be achieved by shifting a given primitive by additive constants. Then the product rule gives

$$\frac{d}{dz}\left(f(z)\exp(-H(z))\right) = f'(z)\exp(-H(z)) - f(z)\exp(-H(z))\frac{f'(z)}{f(z)} = 0.$$

Since $\exp(H(z_0)) = f(z_0)$, we deduce from the path-connectedness of G that $\exp(H(z)) = f(z)$ for all $z \in G$. Thus setting $\log(f) = H$ shows the first assertion.

In order to prove the second statement, it suffices to define $\sqrt[n]{f} : G \to \mathbb{C}$ by $\sqrt[n]{f}(z) = \exp(\frac{1}{n}H(z))$. □

Now we are in a position to show the following auxiliary result on holomorphic functions that omit two values.

Lemma 6.11. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic such that $\{-1,1\} \cap f(G) = \emptyset$. Then there exists a holomorphic function $h : G \to \mathbb{C}$ such that*

$$f = \cos(h).$$

Proof. Note that the function $z \mapsto 1 - f(z)^2$ never vanishes on G. Hence by Corollary 6.10 there exists a holomorphic square-root $g = \sqrt{1-f^2} : G \to \mathbb{C}$, which satisfies in particular

$$(f + ig)(f - ig) = f^2 + g^2 = 1. \tag{12}$$

Thus $(f + ig)$ has no zeros in G and therefore we can write $(f + ig) = e^{ih}$ for some holomorphic function $h = -i\log(f + ig) : G \to \mathbb{C}$. Then by (12) we have $(f - ig) = e^{-ih}$, so that $f = \frac{1}{2}(e^{ih} + e^{-ih}) = \cos(h)$, as claimed. □

With this lemma we can prove the following crucial proposition.

Proposition 6.12. *Let $G \subset \mathbb{C}$ be a simply connected domain and let $f : G \to \mathbb{C}$ be holomorphic such that $\{0,1\} \cap f(G) = \emptyset$. Then there exists a holomorphic function $h : G \to \mathbb{C}$ such that*

$$f = \frac{1}{2}\left(1 + \cos(\pi \cos(\pi h))\right). \tag{13}$$

Furthermore, if $\widetilde{h} : G \to \mathbb{C}$ is any holomorphic function satisfying (13), then $\widetilde{h}(G)$ does not contain any open ball of radius 1.

Proof. First note that the function $2f - 1$ omits the values -1 and 1, so that by Lemma 6.11 we find a holomorphic function $h_1 : G \to \mathbb{C}$ such that $2f - 1 = \cos(\pi h_1)$. Observe further that h_1 must omit all integer values. Hence again by Lemma 6.11 we can write $h_1 = \cos(\pi h)$ for some holomorphic function $h : G \to \mathbb{C}$. The first claim then follows by rearranging terms.

Now let $\widetilde{h} : G \to \mathbb{C}$ be any such function. Define the grid-like set

$$\mathcal{L} = \{m \pm i\pi^{-1} \log(n + \sqrt{n^2 - 1}) : \ m, n \in \mathbb{Z}, n \geq 1\}.$$

We shall prove that $\mathcal{L} \cap \widetilde{h}(G) = \emptyset$. Indeed, for $\hat{z} := m \pm i\pi^{-1} \log(n + \sqrt{n^2 - 1}) \in \mathcal{L}$ we have that

$$\begin{aligned}\cos(\pi\hat{z}) =& \frac{1}{2}(e^{i\pi\hat{z}} + e^{-i\pi\hat{z}}) = \frac{1}{2}(-1)^m \left((n + \sqrt{n^2-1})^{\mp 1} + (n + \sqrt{n^2-1})^{\pm 1}\right) \\ =& \frac{1}{2}(-1)^m \frac{n^2 + n^2 - 1 + 2n\sqrt{n^2-1} + 1}{n + \sqrt{n^2-1}} = (-1)^m n.\end{aligned}$$

Thus $\cos(\pi\cos(\pi\hat{z})) = \pm 1$ for all $\hat{z} \in \mathcal{L}$. Since $f(G) \cap \{0,1\} = \emptyset$ we conclude that $\widetilde{h}(G) \cap \mathcal{L} = \emptyset$.

It remains to estimate the spacing between points of $\mathcal{L}$. Note that the points of $\mathcal{L}$ are contained in the vertical lines $\mathrm{Re}(z) = m$ for $m \in \mathbb{Z}$. Within each vertical line $\mathrm{Re}(z) = m$, the distance between neighboring points of $\mathcal{L}$ is less than 1, since for any integer $n \geq 1$ we have

$$\begin{aligned}\pi^{-1} \cdot \left|\log(n + 1 + \sqrt{(n+1)^2 - 1}) - \log(n + \sqrt{n^2 - 1})\right| &= \pi^{-1} \cdot \left|\log\left(\frac{1 + n^{-1} + \sqrt{1 + 2n^{-1}}}{1 + \sqrt{1 - n^{-2}}}\right)\right| \\ &\leq \frac{\log(1 + n^{-1} + \sqrt{1 + 2n^{-1}})}{\pi} \leq \frac{\log(2 + \sqrt{3})}{\pi} < 1.\end{aligned}$$

Hence for every $z \in \mathbb{C}$ there exists $\hat{z} \in \mathcal{L}$ such that $|\mathrm{Re}(z) - \mathrm{Re}(\hat{z})| \leq 1/2$ and $|\mathrm{Im}(z) - \mathrm{Im}(\hat{z})| < 1/2$, so that $|z - \hat{z}| < \frac{\sqrt{2}}{2} < 1$. Therefore every open ball of radius 1 in $\mathbb{C}$ intersects $\mathcal{L}$. We conclude that $\widetilde{h}(G)$ cannot contain any open ball of radius one.

□

Proposition 6.12 directly implies Picard's little theorem (cf. Exercise H 10.1). For our proof of Picard's great theorem, we will also need to control the growth of functions omitting the two values 0 and 1.

Let $\beta > 0$ denote a constant for which Bloch's theorem holds (e.g. $\beta = 3/2 - \sqrt{2}$). Consider the function $L : (0,1) \times (0,\infty) \to \mathbb{R}_{>0}$ given by

$$L(\theta, r) := \exp\left(\pi \exp\left(\pi(3 + 2r) + \frac{\pi\theta}{\beta(1-\theta)}\right)\right).$$

Then we have the following result giving, for holomorphic functions omitting two values, a uniform bound in a neighborhood of a point assuming only a uniform bound at that point. The precise bound is immaterial (and can be improved).

Theorem 6.13 (Schottky's theorem)**.** *Let $f \in \mathcal{H}(\overline{B_1(0)})$ be such that $|f(0)| \leq r$ and $f(\overline{B_1(0)}) \cap \{0,1\} = \emptyset$. Then*

$$|f(z)| \leq L(\theta, r)$$

whenever $0 < \theta < 1$ and $|z| \leq \theta$.

Proof. We divide the proof into several steps.
Step 1: We show that $\cos(\pi a) = \cos(\pi b)$ if and only if $b = \pm a + 2n$ for some $n \in \mathbb{Z}$, and that for every $w \in \mathbb{C}$ there exists $v \in \mathbb{C}$ with $|v| \leq 1 + |w|$ such that $\cos(\pi v) = w$.

The first claim follows from the elementary formula $\cos(\pi a) - \cos(\pi b) = -2\sin(\frac{\pi}{2}(a+b))\sin(\frac{\pi}{2}(a-b))$, as the set of zeros of $z \mapsto \sin(z)$ is $\pi\mathbb{Z}$. For the second claim, since $z \mapsto \cos(z)$ is surjective onto $\mathbb{C}$, for every $w \in \mathbb{C}$ we can find $v \in \mathbb{C}$ with $\mathrm{Re}(v) \in [-1,1]$ and $\cos(\pi v) = w$. Then the triangle inequality gives

$$|w| = |\cos(\pi v)| = \Big|\frac{e^{i\pi v} + e^{-i\pi v}}{2}\Big| \geq \frac{e^{\pi|\mathrm{Im}(v)|} - 1}{2} \geq \frac{\pi|\mathrm{Im}(v)|}{2},$$

since $e^x \geq 1 + x$ for $x \geq 0$, so

$$|v| = \sqrt{\mathrm{Re}(v)^2 + \mathrm{Im}(v)^2} \leq \sqrt{1 + 4|w|^2/\pi^2} \leq 1 + |w|.$$

Step 2: There exists a function $g \in \mathcal{H}(\overline{B_1(0)})$ such that

(i) $f = \frac{1}{2}(1 + \cos(\pi\cos(\pi g)))$ with $|g(0)| \leq 3 + 2|f(0)|$;
(ii) $|g(z)| \leq |g(0)| + \frac{\theta}{\beta(1-\theta)}$ for all $0 < \theta < 1$ and $|z| \leq \theta$.

Let us show (i). By Lemma 6.11 we find a function $\widetilde{F} \in \mathcal{H}(\overline{B_1(0)})$ such that $2f - 1 = \cos(\pi\widetilde{F})$. Due to Step 1, there exists $b \in \mathbb{C}$ such that $\cos(\pi b) = 2f(0) - 1$ and $|b| \leq 1 + |2f(0) - 1| \leq 2 + 2|f(0)|$. Moreover, again by Step 1 we have $b = \pm\widetilde{F}(0) + 2k$ with $k \in \mathbb{Z}$. Define $F = \pm\widetilde{F} + 2k$, so that $F \in \mathcal{H}(\overline{B_1(0)})$. Then $2f - 1 = \cos(\pi F)$ and $F(0) = b$. Since F omits all integer values (otherwise f would attain either 0 or 1), there exists $\widetilde{g} \in \mathcal{H}(\overline{B_1(0)})$ such that $F = \cos(\pi\widetilde{g})$. Using Step 1 one more time, we find $a \in \mathbb{C}$ such that $\cos(\pi a) = b$ and $|a| \leq 1 + |b| \leq 3 + 2|f(0)|$. By construction $\cos(\pi a) = \cos(\pi\widetilde{g}(0))$, so that we can again define $g = \pm\widetilde{g} + 2m \in \mathcal{H}(\overline{B_1(0)})$ for some $m \in \mathbb{Z}$ such that $g(0) = a$ and $F = \cos(\pi g)$. Then $f = \frac{1}{2}(1 + \cos(\pi\cos(\pi g)))$ and $|g(0)| = |a| \leq 3 + 2|f(0)|$ as claimed in (i).

In order to show (ii), note that by Proposition 6.12 we know that $g(B_1(0))$ contains no open ball of radius 1. Since $\mathrm{dist}(z, \partial B_1(0)) \geq (1-\theta)$ for all $|z| \leq \theta$, the generalized Bloch theorem (Corollary 6.7) implies that $\beta(1-\theta)|g'(z)| \leq 1$ for all $|z| \leq \theta$, so $|g'(z)| \leq (\beta(1-\theta))^{-1}$. Thus by the fundamental theorem of calculus

$$\begin{aligned} |g(z)| \leq |g(z) - g(0)| + |g(0)| &\leq \int_{[0,z]} |g'(\zeta)|\, d\zeta + |g(0)| \\ &\leq \frac{|z|}{\beta(1-\theta)} + |g(0)| \leq \frac{\theta}{\beta(1-\theta)} + |g(0)| \end{aligned}$$

for all $|z| \leq \theta$.
Step 3: Conclusion.

We finish the proof by noting that $|\cos(w)| = \frac{1}{2}|e^{iw} + e^{-iw}| \leq e^{|w|}$ and $\frac{1}{2}|1 + \cos(w)| \leq e^{|w|}$. Using these bounds and properties (i) and (ii) of Step 2 we deduce that if $|z| \leq \theta$ then

$$|f(z)| \leq e^{\pi|\cos(\pi g(z))|} \leq \exp(\pi\exp(\pi|g(z)|)) \leq \exp\Big(\pi\exp\Big(\pi\Big(3 + 2|f(0)| + \frac{\theta}{\beta(1-\theta)}\Big)\Big)\Big) \leq L(\theta, r),$$

where in the last estimate we used that $|f(0)| \leq r$.

□

Schottky's theorem leads to the following sharpened version of Montel's compactness theorem.

Theorem 6.14 (Sharpened version of Montel's theorem)**.** *Let $D \subset \mathbb{C}$ be a domain and*

$$\mathcal{F} := \{f : D \to \mathbb{C} \text{ holomorphic with } \{0,1\} \cap f(D) = \emptyset\}.$$

Then every sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ either contains a subsequence that converges locally uniformly to some holomorphic function $f : D \to \mathbb{C}$, or the whole sequence $\{|f_n|\}_{n\in\mathbb{N}}$ converges locally uniformly to ∞.

Remark 6.15. Local uniform convergence of a sequence $\{g_n\}_{n\in\mathbb{N}}$ to ∞ is defined by requiring that $1/g_n$ converges locally uniformly to 0.

Proof of Theorem 6.14. We divide the proof into several steps.
Step 1: For $z_0 \in D$ and $r \geq 0$, define $\mathcal{F}(z_0, r) := \{f \in \mathcal{F} : |f(z_0)| \leq r\}$. We argue that there exists a neighborhood of z_0 on which $\mathcal{F}(z_0, r)$ is uniformly bounded.

Indeed, choose $\delta > 0$ such that $\overline{B_{2\delta}(z_0)} \subset D$. Applying Schottky's Theorem 6.13 to each function $z \mapsto f(2\delta z + z_0) \in \mathcal{H}(\overline{B_1(0)})$ for $f \in \mathcal{F}(z_0, r)$ we infer that

$$\sup_{f \in \mathcal{F}(z_0,r)} \sup_{z \in B_\delta(z_0)} |f(z)| \leq L(1/2, r).$$

This proves the first step.
Step 2: The family $\mathcal{F}(z_0, r)$ is locally uniformly bounded in D.

Indeed, note that $U := \{z \in D : \mathcal{F}(z_0, r)$ is uniformly bounded in some neighborhood of $z\}$ is open in D. Moreover, $z_0 \in U$ by Step 1. Below we argue that U is also closed in D. Then connectedness of D implies that $U = D$, so the claim of Step 2 follows.

Let $w \in \partial U \cap D$ be such that $\mathcal{F}(z_0, r)$ is not uniformly bounded in any neighborhood of w. If $\sup_{h \in \mathcal{F}(z_0,r)} |h(w)| =: R < \infty$, then we would have $\mathcal{F}(z_0, r) \subset \mathcal{F}(w, R)$, but by Step 1 we know that $\mathcal{F}(w, R)$ is uniformly bounded in some neighborhood of w, contradiction. Therefore there exists a sequence $\{h_n\}_{n \in \mathbb{N}} \subset \mathcal{F}(z_0, r)$ such that $\lim_{n\to\infty} |h_n(w)| = \infty$. Define $g_n = 1/h_n$ (which is holomorphic in D and also omits 0 and 1), so that $\lim_{n\to\infty} g_n(w) = 0$. Hence $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(w, R)$ for some suitable $0 < R < \infty$, so by Step 1 the sequence $\{g_n\}_{n\in\mathbb{N}}$ is uniformly bounded in a neighborhood of w. Therefore by Montel's theorem (Theorem 1.7) we can pass to a subsequence that converges uniformly in some ball $B_t(w)$ to a holomorphic function $g : B_t(w) \to \mathbb{C}$. Since the g_n do not vanish and $g(w) = 0$, it follows from Corollary 1.6 that $g \equiv 0$. Since $B_t(w) \cap U \neq \emptyset$ we conclude that $\limsup_{n\to\infty} |h_n(z)| = \infty$ also for some $z \in U$, which gives a contradiction.
Step 3: Conclusion.

Fix $z_0 \in D$ and let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ be a sequence. If $f_n \in \mathcal{F}(z_0, 1)$ for infinitely many $n \in \mathbb{N}$ then the desired result follows from Step 2 and Montel's theorem (Theorem 1.7). If only finitely many f_n belong to $\mathcal{F}(z_0, 1)$, then infinitely many $g_n = 1/f_n$ belong to $\mathcal{F}(z_0, 1)$. By the same argument we conclude that either the g_n converge locally uniformly to $g \equiv 0$, or some subsequence of the g_n converges locally uniformly to a non-identically zero $g : D \to \mathbb{C}$, which then has no zeros at all by Corollary 1.6. In the second case the f_n then converge locally uniformly (to $1/g$) along that subsequence. In the first case $|f_n| \to \infty$ locally uniformly along the whole sequence, as desired.

□

Now we can finally prove Picard's great theorem.

Proof of Theorem 6.2. We have seen in Exercise H 10.3 the remarkable result that Picard's great theorem is equivalent to the following statement: for any holomorphic function $f : B_1(0) \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$, either f or $1/f$ is bounded in a neighborhood of the origin.

Consider the sequence of holomorphic functions $f_n(z) = f(z/n) : B_1(0) \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$. By Theorem 6.14 there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that either $\{f_{n_k}\}_{k\in\mathbb{N}}$ or $\{1/f_{n_k}\}_{k\in\mathbb{N}}$ is locally uniformly bounded on $B_1(0)\setminus\{0\}$, since one of them must be locally uniformly convergent in this domain. In the first case there exists $0 < C < \infty$ such that

$$\sup_k |f(z/n_k)| \leq C \qquad \text{whenever } |z| = \frac{1}{2}.$$

Hence by the maximum principle $|f(z)| \leq C$ on each annulus $A_k = \{z \in \mathbb{C} : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k}\}$. Since

$$A := \bigcup_{k\in\mathbb{N}} A_k$$

is a (punctured) neighborhood of the origin, we conclude that f is bounded in a neighborhood of the origin. The case when $\{1/f_{n_k}\}_{k\in\mathbb{N}}$ is locally uniformly bounded can be treated by the same argument, which concludes the proof.

□

7. The Riemann mapping theorem

In this chapter we will prove one of the main theorems of complex analysis. The Riemann mapping theorem classifies the open sets which are biholomorphically equivalent to the open unit disc $B_1(0)$. Two open sets $U_1, U_2 \subset \mathbb{C}$ are said to be biholomorphically equivalent if there exists a bijective holomorphic map $f : U_1 \to U_2$ such that the inverse map $f^{-1} : U_2 \to U_1$ is also holomorphic. It follows from the definition that biholomorphic equivalence of open sets is an equivalence relation.

In order to study the family of open sets which are biholomorphically equivalent to the unit disc $B_1(0)$, first note that by Liouville's theorem $\mathbb{C}$ cannot belong to that class. Moreover, as biholomorphic functions are in particular homeomorphisms, all open sets belonging to that class share the same topological invariances. In particular, such open sets have to be path-connected and also simply connected (cf. Definition 6.8).

Simply connected domains in $\mathbb{C}$ can be informally described as having no holes. The surprising fact of the Riemann mapping theorem is that this topological restriction already ensures biholomorphic equivalence to the unit disc. In particular, no regularity of the boundary is required.

In the proof of the Riemann mapping theorem we will use the fact that injective holomorphic functions are already biholomorphic onto their image. For the sake of completeness we include the proof.

Lemma 7.1. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic and injective. Then $f'(z_0) \neq 0$ for all $z_0 \in U$, and the inverse function $f^{-1} : f(U) \to \mathbb{C}$ is also holomorphic.*

Proof. Assume by contradiction that $f'(z_0) = 0$ for some $z_0 \in U$. Upon considering the function $z \mapsto f(z + z_0) - f(z_0)$ we may assume that $z_0 = f(z_0) = f'(z_0) = 0$. Since f is injective it is not identically zero on a neighborhood of $z_0 = 0$, so the order $k := o_0(f) \geq 2$ of the zero of $f(z)$ at $z = 0$ is well-defined. Hence we can write

$$f(z) = z^k g(z)$$

with $g : U \to \mathbb{C}$ holomorphic and $g(0) \neq 0$. Since g is in particular continuous, there exists $r > 0$ such that $B_r(0) \subset U$ and $\inf_{z \in B_r(0)} |g(z)| > 0$. Since $B_r(0)$ is a simply connected domain, we can apply Corollary 6.10 to infer that there exists a holomorphic function $h : B_r(0) \to \mathbb{C}$ such that $h^k = g$. Then for all $z \in B_r(0)$ we have

$$f(z) = (zh(z))^k.$$

Note that the function $z \mapsto zh(z)$ is non-constant and holomorphic. Hence by the open mapping theorem there exists $r_1 > 0$ and $z_1, z_2 \in B_r(0)$ such that $z_1 h(z_1) = r_1$ and $z_2 h(z_2) = r_1 \exp(2\pi i/k) \neq r_1$. This contradicts the injectivity of f since $f(z_1) = f(z_2)$. Hence $f'(z_0) \neq 0$ for all $z_0 \in U$.

Now, by the open mapping theorem it follows that $f(U)$ is open. Let $z_0 \in U$ and set $w_0 = f(z_0)$. For any $w \in f(U)$, write $w = f(z)$ for some $z \in U$. Denoting $g = f^{-1} : f(U) \to U$, the open mapping theorem (for f) shows that g is continuous. We have

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}.$$

Since $f'(z_0) \neq 0$, taking the limit as $w \to w_0$ (so that $z \to z_0$ by continuity of g) yields $g'(w_0) = 1/f'(z_0)$, so g is holomorphic at any $w_0 \in f(U)$.

□

The remainder of this chapter will be about the proof of the Riemann mapping theorem. In the proof we will apply the Schwarz lemma, which we now recall.

Lemma 7.2 (Schwarz lemma)**.** *Let $f : B_1(0) \to \overline{B_1(0)}$ be a holomorphic function such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in B_1(0)$ and $|f'(0)| \leq 1$. Moreover, if equality is achieved in either inequality for some $z \in B_1(0) \setminus \{0\}$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.*

Proof. Consider the decomposition $f(z) = zg(z)$. Since $f(0) = 0$ we know that $g : B_1(0) \to \mathbb{C}$ is holomorphic and $g(0) = f'(0)$. For $0 < r < 1$ and $z \in \partial B_r(0)$ we have

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

Due to the maximum principle, this inequality also holds for all $z \in B_r(0)$. Letting $r \uparrow 1$ yields that $|g(z)| \le 1$ for all $z \in B_1(0)$. This implies $|f(z)| \le |z|$ and $|f'(0)| \le 1$. If either of them is an equality we deduce that $|g(z_0)| = 1$ for some $z_0 \in B_1(0)$. Again by the maximum principle it follows that g is constant. Hence $g(z) = a$ for some $a \in \mathbb{C}$ with $|a| = 1$. This yields the claim. $\square$

Theorem 7.3 (Riemann mapping theorem)**.** *Let $G \subsetneq \mathbb{C}$ be a simply connected domain. Then there exists a biholomorphic map $f : G \to B_1(0)$.*

Proof. Due to Lemma 7.1 it suffices to find a bijective holomorphic map $f : G \to B_1(0)$. We will prove the existence of such a map in three steps.

(1) We show the existence of an injective holomorphic map $g : G \to B_1(0)$ with $0 \in g(G)$. This allows us to assume that $G \subset B_1(0)$ and $0 \in G$;
(2) For $0 \in G \subset B_1(0)$ and an injective holomorphic map $f : G \to B_1(0)$ with $f(0) = 0$, we show that surjectivity would follow from the maximality of $|f'(0)|$ among all such maps;
(3) We show the existence of a map f as in the previous item and with $|f'(0)|$ maximal among all such maps (i.e. attaining the supremum).

The desired function can then be obtained as the composition $f \circ g$.

Step 1: Assume for the moment that the complement $\mathbb{C} \setminus G$ contains an open ball $B_{2r}(z_0)$. Then the map $g_1 : G \to B_1(0)$ given by $g_1(z) = r(z - z_0)^{-1}$ is well-defined, holomorphic and injective. Let $z_1 \in g(G) \subset \overline{B_{1/2}(0)}$. Then the map $g(z) = \frac{1}{2}(g_1(z) - z_1)$ is still injective, holomorphic and $0 \in g(G) \subset B_1(0)$. This gives the desired map of step 1.

However, in general we cannot assume that $\mathbb{C}\setminus G$ contains an open ball. Therefore we have to transform it via an injective, holomorphic function. The idea is to use a holomorphic square-root. By assumption $G \neq \mathbb{C}$, so there exists $z_0 \in \mathbb{C} \setminus G$. Then the function $z \mapsto z - z_0$ never vanishes on G. Hence by Corollary 6.10 there exists a holomorphic function $G \ni z \mapsto \sqrt{z - z_0}$. We claim that this square-root is injective. Indeed, if $\sqrt{z_1 - z_0} = \sqrt{z_2 - z_0}$, then by definition

$$z_1 - z_0 = \left(\sqrt{z_1 - z_0}\right)^2 = \left(\sqrt{z_2 - z_0}\right)^2 = z_2 - z_0,$$

which implies that $z_2 = z_1$. Moreover, we argue that $\mathbb{C} \setminus \sqrt{G - z_0}$ contains an interior point. By the open mapping theorem there exists $\hat{z} \neq 0$ and $r > 0$ such that $B_{2r}(\hat{z}) \subset \sqrt{G - z_0}$. We claim that $-B_{2r}(\hat{z}) \subset \mathbb{C} \setminus \sqrt{G - z_0}$. Indeed, assume that there exists $z_1, z_2 \in G$ such that $\sqrt{z_1 - z_0} = -\sqrt{z_2 - z_0}$. Taking the square yields $z_1 = z_2$, which then gives $\sqrt{z_1 - z_0} = 0$, contradicting $z_0 \notin G$. Thus we are in a position to apply the first part of this step to conclude the proof.

From now on we can assume that $0 \in G \subset B_1(0)$. For this reduction we also used that the image of G under an injective holomorphic function is still simply connected (see Exercise H 11.1).

Step 2: We claim that if $0 \in G \subset B_1(0)$ and $f : G \to B_1(0)$ is holomorphic, injective, satisfies $f(0) = 0$, but fails to be surjective, then there exists a holomorphic, injective function $\widetilde{f} : G \to B_1(0)$ with $\widetilde{f}(0) = 0$ and $|\widetilde{f}'(0)| > |f'(0)|$.

As a first step, note that for any $z_0 \in B_1(0)$ the map $\varphi_{z_0} : B_1(0) \to \mathbb{C}$ defined by

$$\varphi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}$$

is a biholomorphic map onto $B_1(0)$ (cf. Exercise H 9.2). If f is not surjective, there exists $z_0 \in B_1(0)\setminus f(G)$. Then $\varphi_{z_0} \circ f : G \to B_1(0)$ satisfies $\varphi_{z_0}(f(z)) \neq 0$ for all $z \in G$. By Corollary 6.10 we can define a holomorphic square-root of this map. Then the function $\sqrt{\varphi_{z_0} \circ f}$ is holomorphic, injective and $z_1 := \sqrt{\varphi_{z_0}(f(0))} = \sqrt{-z_0} \in B_1(0)$. We define a competitor for f as

$$\widetilde{f} := \varphi_{z_1} \circ \sqrt{\varphi_{z_0} \circ f} : G \to B_1(0),$$

which is injective, holomorphic and satisfies $\widetilde{f}(0) = 0$. Denoting $s(z) = z^2$, the map $h : B_1(0) \to B_1(0)$ given by $h = \varphi_{z_0}^{-1} \circ s \circ \varphi_{z_1}^{-1}$ is holomorphic with $h(0) = 0$ and $h \circ \widetilde{f} = f$. By the Schwarz lemma we know that $|h'(0)| < 1$, since h is not a pure rotation (it is not even injective). Hence the chain rule implies

$$|f'(0)| = \left|h'(\widetilde{f}(0)) \cdot \widetilde{f}'(0)\right| = \left|h'(0) \cdot \widetilde{f}'(0)\right| < \left|\widetilde{f}'(0)\right|,$$

as claimed.

Step 3: In order to construct a bijective holomorphic function $f : G \to B_1(0)$, it is enough to find f which attains the supremum

$$\sup\{|f'(0)| : \text{the function } f : G \to B_1(0) \text{ is holomorphic and injective with } f(0) = 0\}.$$

Indeed, by Step 2 such a function is surjective and therefore satisfies the claimed properties. We have already seen in Exercise H 3.3 (using Montel's theorem) that a maximizer for the above extremal problem exists, provided that the class of competitors is not empty. Since we reduced the analysis to the case that $0 \in G \subset B_1(0)$, the function $z \mapsto z$ is admissible. This concludes the proof.

□

Remark 7.4. One can prove that under the assumption that $f(z_0) = 0$ and $f'(z_0) \in (0, \infty)$ for some $z_0 \in G$, the biholomorphic map $f : G \to B_1(0)$ is unique. Indeed, suppose there is another biholomorphic map $g : G \to B_1(0)$ with the given properties. Then the biholomorphic map $h := f \circ g^{-1} : B_1(0) \to B_1(0)$ satisfies $h(0) = 0$ and $h'(0) = f'(z_0)/g'(z_0) \in (0, \infty)$. The inverse map h^{-1} also satisfies the given properties. Applying the Schwarz lemma to both h and h^{-1} yields $|h(z)| = |z|$ for all $z \in B_1(0)$. Hence again by the Schwarz lemma $h(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$. Then from $h'(0) = a \in (0, \infty)$ we infer that $a = 1$, hence $f = g$. This proves uniqueness.

We will not discuss whether the map f given by the Riemann mapping theorem can be extended to the boundary (i.e. to a continuous bijection from $\overline{G}$ to $\overline{B_1(0)}$). Clearly that requires G to be bounded, and in that case such an extension exists if and only if the boundary ∂G is a Jordan curve (i.e. homeomorphic to the unit circle $\partial B_1(0)$). This is the content of Carathéodory's theorem.

8. Holomorphic functions on the Riemann sphere

In many situations it is convenient to allow the value ∞ either in the domain or the image of functions (e.g. for meromorphic functions, or in the sharpened version of Montel's theorem). This can be done via the one-point compactification of $\mathbb{C}$, denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Open sets in $\widehat{\mathbb{C}}$ are defined to be exactly those sets $\widehat{U} \in \widehat{\mathbb{C}}$ such that

(i) $\widehat{U}$ is open in $\mathbb{C}$, if $\infty \notin \widehat{U}$;
(ii) $\widehat{U} \setminus \{\infty\} = \mathbb{C} \setminus K$ for some compact set $K \subset \mathbb{C}$, if $\infty \in \widehat{U}$.

With this topology one can prove that $\widehat{\mathbb{C}}$ is a compact metrizable space which is homeomorphic to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ (cf. Exercise H 12.1). Moreover, a sequence $\{z_n\}_{n\in\mathbb{N}}$ converges to ∞ if and only if $1/z_n$ converges to zero, while the convergence on $\widehat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$ remains unaffected. In this sense, we do not distinguish "in which direction" the sequence approaches infinity. Here and in what follows we set $1/\infty = 0$ and $1/0 = \infty$. Then we have the following definition of holomorphic functions $f : \widehat{U} \to \widehat{\mathbb{C}}$.

Definition 8.1. Let $\widehat{U} \subset \widehat{\mathbb{C}}$ be open and let $f : \widehat{U} \to \widehat{\mathbb{C}}$ be continuous. Then f is called complex differentiable at $z_0 \in \widehat{U}$ if

(i) f is complex differentiable at z_0 in the usual sense, if $z_0 \in \mathbb{C}$ and $f(z_0) \in \mathbb{C}$;
(ii) $g(z) = f\left(\frac{1}{z}\right)$ is complex differentiable at 0, if $z_0 = \infty$ and $f(z_0) \in \mathbb{C}$;
(iii) $g(z) = \frac{1}{f(z)}$ is complex differentiable at z_0, if $z_0 \in \mathbb{C}$ and $f(z_0) = \infty$;
(iv) $g(z) = \frac{1}{f\left(\frac{1}{z}\right)}$ is complex differentiable at 0, if $z_0 = f(z_0) = \infty$.

Moreover, f is called holomorphic on $\widehat{U}$ if it is complex differentiable at every point $z_0 \in \widehat{U}$.

Remark 8.2. In the theory of Riemann surfaces the above notion corresponds to holomorphic functions on a complex manifold, since $z \mapsto z$ and $z \mapsto 1/z$ are local charts for the one-dimensional complex manifold $\widehat{\mathbb{C}}$.

Similarly to the case of domains $D \subset \mathbb{C}$, the identity theorem also holds for holomorphic functions $f : \widehat{D} \to \widehat{\mathbb{C}}$, where domains $\widehat{D}$ in $\widehat{\mathbb{C}}$ are defined to be open, path-connected subsets.

Theorem 8.3 (Identity theorem)**.** *Let $\widehat{D} \subset \widehat{\mathbb{C}}$ be a domain and let $f, g : \widehat{D} \to \widehat{\mathbb{C}}$ be holomorphic. If the set $\{f = g\}$ has an accumulation point in $\widehat{D}$, then $f = g$.*

Proof. Let us define $S := \{z \in \widehat{D} : f = g$ in a neighborhood of $z\}$. We argue that $S = \widehat{D}$. First note that S is open. Next we show that $S \neq \emptyset$. To this end, let $z_0 \in \widehat{D}$ be an accumulation point of $\{f = g\}$. Then by continuity $f(z_0) = g(z_0)$. We apply the classical identity theorem to one of the following four holomorphic functions, with a suitably small $r > 0$:

(i) $f, g : B_r(z_0) \to \mathbb{C}$ when $z_0 \in \mathbb{C}$ and $f(z_0) \in \mathbb{C}$;
(ii) $1/f, 1/g : B_r(z_0) \to \mathbb{C}$ when $z_0 \in \mathbb{C}$ and $f(z_0) = \infty$;
(iii) $B_r(0) \ni z \mapsto f\left(\frac{1}{z}\right), g\left(\frac{1}{z}\right)$ when $z_0 = \infty$ and $f(z_0) \in \mathbb{C}$;
(iv) $B_r(0) \ni z \mapsto \frac{1}{f\left(\frac{1}{z}\right)}, \frac{1}{g\left(\frac{1}{z}\right)}$ when $z_0 = f(z_0) = \infty$.

In all four cases we deduce from the classical identity theorem that $f = g$ in a neighborhood of z_0. It thus remains to show that S is also closed. Then connectedness of $\widehat{D}$ yields that $S = \widehat{D}$. Consider a point $s_0 \in \widehat{D}$ such that there exists a sequence $\{s_n\}_{n \in \mathbb{N}} \subset S$ with $s_n \to s_0$. If $s_0 \notin S$ then s_0 is also an accumulation point of $\{f = g\}$, so the argument above shows that $s_0 \in S$, which yields a contradiction. This concludes the proof.

□

One can show that holomorphic functions $f : \widehat{D} \to \widehat{\mathbb{C}}$ which are not constantly ∞ can be identified with meromorphic functions on $\widehat{D} \setminus \{\infty\}$, since by the previous theorem the set $f^{-1}(\infty)$ has no accumulation points in $\widehat{D}$. We will show that holomorphic functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are exactly rational functions. To this end, we first prove some auxiliary results with a similar flavor.

Lemma 8.4. *Let $f : \widehat{\mathbb{C}} \to \mathbb{C}$ be holomorphic. Then f is constant.*

Proof. Since $\widehat{\mathbb{C}}$ is compact (see Exercise H 12.1) and f is continuous, it follows that $f(\widehat{\mathbb{C}})$ is compact. Hence $f(\mathbb{C}) \subset f(\widehat{\mathbb{C}}) \subset \mathbb{C}$ is bounded, so by Liouville's theorem $f|_{\mathbb{C}}$ is constant. By continuity of f at ∞ we conclude that f is constant on $\widehat{\mathbb{C}}$.

□

When $P : \mathbb{C} \to \mathbb{C}$ is a non-constant polynomial, then one can check that $P(\infty) := \infty$ gives a holomorphic extension $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Our next result states that polynomials are the only functions for which such an extension exists.

Lemma 8.5. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic and such that $f(z) \in \mathbb{C}$ for all $z \in \mathbb{C}$. Then $f|_{\mathbb{C}}$ is a polynomial.*

Proof. See Exercise H 12.3.

□

The next theorem provides a complete characterization of holomorphic functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Theorem 8.6. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic. Then there exist polynomials $P, Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

$$f(z) = \frac{P(z)}{Q(z)} \qquad \forall z \in \mathbb{C} \setminus f^{-1}(\infty).$$

Proof. See Exercise H 12.4.

□

Remark 8.7. Without loss of generality we can assume that P and Q have no common zeros. Then $f(z) = \frac{P(z)}{Q(z)}$ holds for all $z \in \widehat{\mathbb{C}}$, in the sense that the fraction ∞/∞ (possibly arising at $z = \infty$) has to be interpreted depending on the degrees (and possibly the leading coefficients, if $\deg(f) = \deg(g)$) of P

and Q. In that sense one can also show that every rational function (with P and Q not both identically zero) is holomorphic on $\widehat{\mathbb{C}}$.

With the above theorem we can easily identify the biholomorphic functions from $\widehat{\mathbb{C}}$ to itself.

Corollary 8.8. *A function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is biholomorphic if and only if f is a so-called Möbius transformation, i.e. there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ and*

$$f(z) = \frac{az + b}{cz + d},$$

with the convention that $f(\infty) = a/c \in \widehat{\mathbb{C}}$ and $f(-d/c) = \infty$.

Proof. First note that if f is of the above type then f is rational, hence holomorphic on $\widehat{\mathbb{C}}$. It has its only pole at $z = -d/c$ (note that the condition $ad - bc \neq 0$ rules out that pole being canceled). A direct calculation shows that its inverse is given by

$$f^{-1}(z) = \begin{cases} \frac{dz-b}{-cz+a} & \text{if } z \in \mathbb{C} \setminus \{a/c\}, \\ \infty & \text{if } z = a/c \in \widehat{\mathbb{C}}, \\ -\frac{d}{c} & \text{if } z = \infty. \end{cases}$$

This is again a rational function, so that it is holomorphic on $\widehat{\mathbb{C}}$. Thus f is biholomorphic.

Now we prove the converse statement. If $f(\infty) = \infty$, then $f(\mathbb{C}) \subset \mathbb{C}$ and by Exercise H 11.4 b) we know that f is affine, which yields the desired result for $a \neq 0$, $b \in \mathbb{C}$, $c = 0$, $d = 1$. Hence assume without loss of generality that $f(\infty) \in \mathbb{C}$. Composing f with the Möbius transformation

$$\varphi(z) = \frac{1}{z - f(\infty)},$$

we obtain a biholomorphic function $\tilde{f} = \varphi \circ f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\tilde{f}(\infty) = \infty$. Hence again by Exercise H 11.4 b) we know that $\tilde{f}$ is affine, so there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $\tilde{f}(z) = az + b$ for all $z \in \mathbb{C}$. Then by the first part of the proof

$$f(z) = (\varphi^{-1} \circ \tilde{f})(z) = \frac{-f(\infty)(az + b) - 1}{-(az + b)} = \frac{-f(\infty)az + (-1 - f(\infty)b)}{-az - b}$$

is a Möbius transformation, since $f(\infty)ab - a(1 + f(\infty)b) = -a \neq 0$. From a more abstract point of view, we used that the Möbius transformations form a group with respect to composition, in fact isomorphic to $\mathrm{PGL}_2(\mathbb{C})$.

□

We stop here with our short introduction to holomorphic functions on the Riemann sphere. Further details should be studied from the more general viewpoint of Riemann surfaces.

9. An introduction to complex analysis in several variables

In the final chapter of the course we briefly discuss holomorphic functions $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}^n$ is open and $n \geq 2$. This introduction is by no means complete.

First let us define what we mean by a holomorphic function in several variables. In what follows we let $\| \cdot \|$ be any norm on $\mathbb{C}^n$ (recall that all norms on a finite dimensional vector space are equivalent).

Definition 9.1. Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. Then f is called complex differentiable at $a \in U$ if there exists a $\mathbb{C}$-linear map $Df(a) : \mathbb{C}^n \to \mathbb{C}$ such that

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{|f(a + h) - f(a) - Df(a)h|}{\|h\|} = 0.$$

Furthermore, f is called holomorphic on U if it is complex differentiable at every $a \in U$. More generally, a function $f : U \to \mathbb{C}^m$ is called holomorphic if each component is holomorphic.

Remark 9.2. As in the theory of one complex variable, there are several equivalent definitions of holomorphic functions $f : U \to \mathbb{C}$, such as the following.

(i) The function f is holomorphic in each variable separately, i.e. for each fixed $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ the function $z \mapsto f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots z_n)$ is holomorphic on the open set
$$U(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) := \{z \in \mathbb{C} : (z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n) \in U\}.$$

(ii) The function f is C^1 (as a map from an open subset of $\mathbb{R}^{2n}$ to $\mathbb{R}^2$) and satisfies the Cauchy-Riemann equations
$$\frac{\partial}{\partial \bar{z}_j} f \equiv 0,$$
where $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$ for $z = x + iy$ with $x, y \in \mathbb{R}^n$ (see Exercise H 13.1 for a proof).

(iii) For each $a \in U$ there exists $r > 0$ such that on $B_r(a)$ the function f can be written as an absolutely convergent power series
$$f(z) = \sum_{\alpha} c_\alpha (z - a)^\alpha,$$
where $\alpha \in (\mathbb{N}_0)^n$ stands for a multi-index and $(z - a)^\alpha := \prod_{1 \le j \le n} (z_j - a_j)^{\alpha_j}$.

(iv) The function f is continuous in each complex variable separately and locally bounded. Moreover, for any $w \in U$ there exists $r > 0$ such that $\overline{D_r(w)} \subset U$ and for all $z \in D_r(w)$ it holds that
$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - w_n| = r} \cdots \int_{|\zeta_1 - w_1| = r} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \dots \, d\zeta_n,$$
where $D_r(w) := \{z \in \mathbb{C}^n : |z_i - w_i| < r \ \forall i\}$ is the so-called polydisc.

One can show that those four conditions are all equivalent to Definition 9.1. There is however a subtle point in this statement. The equivalence of all definitions under the extra assumption that f is locally bounded is elementary, and follows from the first step of the proof of Theorem 9.20 below. While Definition 9.1, (iii), and (iv) each imply local boundedness (and even continuity) of f, this is not clear from (i) or (ii). For (ii) it follows from standard real analysis (and the equivalence is worked out in Exercise H 13.1). For (i) it is a deep result of Hartogs that separately holomorphic functions are continuous (and therefore analytic). We prove this in Theorem 9.20, and refer to [2, Section 2.4] for a more complete discussion of surrounding topics.

Since complex differentiable functions are automatically differentiable (in the real sense as functions from $\mathbb{R}^{2n}$ to $\mathbb{R}^2$), the differential $Df(z_0)$ is unique, linear in f, and the chain rule holds.

Next we will introduce a technique that allows us to transfer some results on holomorphic functions in one complex variable to the multivariable case. This is the so-called method of slicing, which is also used in the calculus of variations.

Lemma 9.3. *Let $U \subset \mathbb{C}^n$ be open and $a \in U$. For $\xi \in \mathbb{C}^n$ define the set $U_{a,\xi}$ by*
$$U_{a,\xi} = \{t \in \mathbb{C} : a + t\xi \in U\}.$$
Given a holomorphic function $f : U \to \mathbb{C}$ we define $f_{a,\xi} : U_{a,\xi} \to \mathbb{C}$ by
$$f_{a,\xi}(t) = f(a + t\xi).$$
Then $U_{a,\xi} \subset \mathbb{C}$ is open with $0 \in U_{a,\xi}$ and $f_{a,\xi}$ is holomorphic on $U_{a,\xi}$.

Proof. Clearly $a \in U$ implies $0 \in U_{a,\xi}$. Note that $U_{a,\xi}$ is open as the preimage of the open set U under the continuous map $t \mapsto a + t\xi$. By the chain rule, $f_{a,\xi}$ is holomorphic on $U_{a,\xi}$. □

Corollary 9.4. *We have the analogues of the following results from the single-variable theory:*

1. **Liouville's theorem:** *Every bounded entire function $f : \mathbb{C}^n \to \mathbb{C}$ is constant.*
2. **Identity theorem:** *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. If $f|_{B_r(a)} \equiv 0$ for some $a \in D$ and $r > 0$, then $f \equiv 0$.*

3. **Open mapping theorem:** *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be non-constant and holomorphic. Then $f(D)$ is again a domain.*
4. **Maximum principle:** *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. If $|f|$ attains its maximum on D then f is constant.*

Proof. See Exercise H 13.2. ☐

By considerations similar to the slicing method, we now prove a multivariable version of the Cauchy integral formula (see also Remark 9.2 (iv)).

In what follows, given a vector $r = (r_1, \dots, r_n) \in (0, \infty)^n$ and $a \in \mathbb{C}^n$ we define the polydisc $D_r^n(a)$ by

$$D_r^n(a) := \{z \in \mathbb{C}^n : |z_i - a_i| < r_i\}.$$

Then we have the following result.

Theorem 9.5 (Cauchy's integral formula for polydiscs)**.** *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be holomorphic. Let $a \in U$ and $r \in (0, \infty)^n$ be such that $\overline{D_r^n(a)} \subset U$. Then for all $z \in D_r^n(a)$ it holds that*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - a_n| = r_n} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta)}{\prod_{i=1}^n (\zeta_i - z_i)} \, d\zeta.$$

Proof. We prove the statement by induction on n. For $n = 1$ the claim coincides with Theorem 0.2, so there is nothing to prove. Next assume that $n \geq 2$. Given $z \in D_r^n(a)$ it follows that $z_n \in B_{r_n}(a_n)$ and $(z_1, \dots, z_{n-1}) \in D^{n-1}_{(r_1, \dots, r_{n-1})}(a_1, \dots, a_{n-1})$, whose closure is contained in the open set $U_{n-1} = \{z' \in \mathbb{C}^{n-1} : (z', z_n) \in U\}$. Hence by the induction hypothesis we can write

$$f(z) = \frac{1}{(2\pi i)^{n-1}} \int_{|\zeta_{n-1} - a_{n-1}| = r_{n-1}} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, \dots, \zeta_{n-1}, z_n)}{\prod_{i=1}^{n-1} (\zeta_i - z_i)} \, d\zeta_1 \dots d\zeta_{n-1}.$$

Note that for fixed $\zeta_1, \dots, \zeta_{n-1}$ in the domain of integration, the function $z \mapsto f(\zeta_1, \dots, \zeta_{n-1}, z)$ is holomorphic on $B_{r_n}(a_n)$ and $\overline{B_{r_n}(a_n)} \subset U' := \{z \in \mathbb{C} : (\zeta_1, \dots, \zeta_{n-1}, z) \in U\}$. Hence we can apply the single-variable result, and the claim follows from Fubini's theorem. ☐

As in the case of one complex variable, Cauchy's integral formula implies analogues of several results from the single-variable theory (with almost identical proofs) that we list below. The detailed arguments can be found for instance in [3], and are contained in the initial steps of our proof of Theorem 9.20 below.

Corollary 9.6 (Higher dimensional consequences of the Cauchy integral formula)**.** *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be holomorphic. Then*

(i) $f \in C^\infty(U)$ and all of its (complex) partial derivatives are holomorphic. Moreover, in the situation of Theorem 9.5, for every multi-index $\alpha \in (\mathbb{N}_0)^n$ it holds that

$$D^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{|\zeta_n - a_n| = r_n} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta)}{(\zeta - z)^{\alpha + \overline{1}}} \, d\zeta,$$

where $\overline{1} = (1, \dots, 1) \in (\mathbb{N}_0)^n$. In particular,

$$|D^\alpha f(a)| \leq \frac{\alpha!}{r^\alpha} \sup_{z \in D_r^n(a)} |f(z)|.$$

(ii) f is analytic, that is for every $w \in U$ there exists an open neighborhood V of w such that for $z \in V$ we have

$$f(z) = \sum_{\alpha \in (\mathbb{N}_0)^n} \frac{D^\alpha f(w)}{\alpha!} (z - w)^\alpha.$$

Moreover, the series converges uniformly on every polydisc $D_r^n(w)$ such that $\overline{D_r^n(w)} \subset U$.

Remark 9.7 (Montel's theorem)**.** Using the bound of Corollary 9.6 (i), one can prove that Montel's theorem (in the version of Chapter 1) also holds in the multivariable case. Indeed, the bound implies that a locally uniformly bounded sequence $f_n : U \subset \mathbb{C}^n \to \mathbb{C}$ is locally equicontinuous. Then the rest of the proof of Montel's theorem (corresponding to the Arzelà–Ascoli theorem) remains unchanged. Furthermore, the local uniform limit of holomorphic functions is still holomorphic. This can be shown using the local uniform convergence on slices and the fact that a function is holomorphic if and only if it is holomorphic in each variable. Here we rely on Hartogs's theorem in a much simpler setting (called Osgood's lemma), because we know a priori that the limit is continuous.

Until now we saw that several properties still hold in the multi-dimensional setting. Next we point out some significant differences.

Remark 9.8 (Some of the differences to the one-variable setting)**.** Let $n \geq 2$.

- We will prove below that isolated singularities of holomorphic functions in $\mathbb{C}^n$ are always removable. Moreover, there cannot be isolated zeros.
- The item above will be a consequence of an extension result which in a more general form reads as follows: let $U \subset \mathbb{C}^n$ be open, let $K \subset U$ be compact, and assume that $U \setminus K$ is connected. If $f : U \setminus K \to \mathbb{C}$ is holomorphic, then f can be extended to a holomorphic function on U. For a proof see [6, Chapter 7, Theorem 3.3] or [7, Theorem 5.4.4].
- We will also prove that the simply connected sets $B_1(0)$ and $D_1^n(0)$ are not biholomorphically equivalent, which rules out an analogue of the Riemann mapping theorem.

Let us formulate the aforementioned extension result in a special situation.

Theorem 9.9 (Special case of Hartogs's extension theorem)**.** *Let $D \subset \mathbb{C}^{n-1}$ be a domain and $A(r,R) := \{z \in \mathbb{C} : r < |z| < R\}$ with $0 \leq r < R \leq \infty$. Let $f : D \times A(r,R) \to \mathbb{C}$ be holomorphic. Assume that there exist $a \in D$ and $\varepsilon > 0$ such that f can be extended holomorphically to $B_\varepsilon(a) \times B_R(0)$. Then f can be extended holomorphically to $D \times B_R(0)$.*

Proof. Given $r < \rho < R$, for $(z', z_n) \in D \times B_\rho(0)$ we define the function

$$f_\rho(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(z', \zeta)}{\zeta - z_n} \, d\zeta.$$

This function is continuous on $D \times B_\rho(0)$ and separately holomorphic in each variable. By Hartogs's theorem (cf. Remark 9.2 (i) or Theorem 9.20, but once again the continuity is guaranteed a priori here, so the result is much easier) it is therefore holomorphic on $D \times B_\rho(0)$. By assumption there exist $a \in D$ and $\varepsilon > 0$ such that f can be extended holomorphically to the set $B_\varepsilon(a) \times B_R(0)$. By the single-variable Cauchy integral formula, it holds that $f = f_\rho$ on $B_\varepsilon(a) \times B_\rho(0)$. Next note that the set $D \times A(r,\rho)$ is a domain in $\mathbb{C}^n$, so by the identity theorem we deduce that $f = f_\rho$ on $D \times A(r,\rho)$. Then the function

$$F(z', z_n) = \begin{cases} f(z', z_n) & \text{if } (z', z_n) \in D \times A(r,R), \\ f_\rho(z', z_n) & \text{if } (z', z_n) \in D \times B_\rho(0) \end{cases}$$

is well-defined, holomorphic, and extends f.

□

Corollary 9.10. *Let $n \geq 2$.*

(i) If $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, then f can be extended to a holomorphic function $f : U \to \mathbb{C}$.
(ii) If $K \subset \mathbb{C}^n$ is compact and such that $\mathbb{C}^n \setminus K$ is connected, then every holomorphic function $f : \mathbb{C}^n \setminus K \to \mathbb{C}$ can be extended to an entire function.
(iii) If $f : U \to \mathbb{C}$ is holomorphic, then f cannot have an isolated zero.
(iv) If $f : \mathbb{C}^n \to \mathbb{C}$ is entire, then $\{f = 0\}$ is either empty or unbounded.

Proof. See Exercise H 14.3.

□

We now rule out an analogue of the Riemann mapping theorem in $\mathbb{C}^n$ for $n \geq 2$. In the proof we will use the following auxiliary lemma, which cannot be deduced from an open mapping theorem (since such a theorem does not hold for vector-valued holomorphic functions, see Exercise H 13.3).

Lemma 9.11. *Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}^m$ be holomorphic (that is, each component $f_k : D \to \mathbb{C}$ is holomorphic for $1 \leq k \leq m$). If $\|f\|_2$ is constant, then f is constant. Here $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{C}^m$.*

Proof. Let us apply the differential operator $\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$ to the equality $\|f(z)\|_2^2 = c$. By the product rule we deduce that

$$0 = \sum_{k=1}^m \frac{\partial}{\partial \overline{z_j}}(f_k(z)\overline{f_k(z)}) = \sum_{k=1}^m \frac{\partial f_k(z)}{\partial \overline{z_j}}\overline{f_k(z)} + f_k(z)\frac{\partial \overline{f_k}(z)}{\partial \overline{z_j}} = \sum_{k=1}^m f_k(z)\frac{\partial \overline{f_k}(z)}{\partial \overline{z_j}},$$

where we used that $\frac{\partial g}{\partial \overline{z_j}} = 0$ for every holomorphic function g (see Remark 9.2 (ii)). Now consider the differential operator $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$. Note that by definition

$$\frac{\partial \overline{f}}{\partial \overline{z_j}} = \overline{\left(\frac{\partial f}{\partial z_j}\right)} \quad \text{and} \quad \frac{\partial \overline{f}}{\partial z_j} = \overline{\left(\frac{\partial f}{\partial \overline{z_j}}\right)}.$$

It is a well-known fact (and follows from a direct computation) that due to the Cauchy-Riemann equations, $\frac{\partial g}{\partial z_j}$ agrees with the complex partial derivative for holomorphic g. Hence taking another partial derivative $\frac{\partial}{\partial z_j}$ we conclude that

$$0 = \sum_{k=1}^m \frac{\partial}{\partial z_j}\left(f_k(z)\frac{\partial \overline{f_k}(z)}{\partial \overline{z_j}}\right) = \sum_{k=1}^m \left|\frac{\partial f_k(z)}{\partial z_j}\right|^2 + f_k(z)\overline{\left(\frac{\partial^2 f_k}{\partial \overline{z_j}\partial z_j}\right)} = \sum_{k=1}^m \left|\frac{\partial f_k(z)}{\partial z_j}\right|^2,$$

where we used that the partial derivatives $\frac{\partial f_k}{\partial z_j}$ of f_k are still holomorphic. Since D is connected, the equation above implies that all f_k are constant. Hence f is constant.

□

Finally, we prove that in several complex variables the analogue of the Riemann mapping theorem does not hold.

Theorem 9.12 (Failure of the Riemann mapping theorem in several variables)**.** *Let $n \geq 2$. Then there exists no biholomorphic map $f : D_1^n(0) \to B_1(0)$, where the ball $B_1(0)$ is defined with respect to the Euclidean metric.*

Remark 9.13. Since both $D_1^n(0)$ and $B_1(0)$ are convex, they are simply connected. Hence the above result indeed shows that the Riemann mapping theorem cannot hold. See [7, Exercise 3.2.3] for an example of a bijective function $\varphi : D_1^2(0) \to B_1(0)$ such that φ and φ^{-1} are real-analytic.

Proof of Theorem 9.12. Assume that there exists a biholomorphic function $f : D_1^n(0) \to B_1(0)$. For any $w \in D_1^1(0) \subset \mathbb{C}$, define a map $F_w : D_1^{n-1}(0) \to \mathbb{C}^n$ by

$$F_w(z') = \frac{\partial f}{\partial z_n}(z', w).$$

We will prove that F_w can be extended continuously to $\partial D_1^{n-1}(0)$ by 0. To this end, take a sequence $\{z'_j\}_{j\in\mathbb{N}} \subset D_1^{n-1}(0)$ such that $\lim_{j\to\infty} z'_j \in \partial D_1^{n-1}(0)$ and define the sequence $f_j : D_1^1(0) \to B_1(0)$ by $f_j(w) = f(z'_j, w)$. By Montel's theorem there exists a subsequence f_j (after relabeling) such that $f_j \to g$ locally uniformly on $D_1^1(0)$, for some holomorphic function $g : D_1^1(0) \to \overline{B_1(0)}$. By construction, for every $w \in D_1^1(0)$ the sequence $\{(z'_j, w)\}_{j\in\mathbb{N}}$ converges to a point $z_w \in \partial D_1^n(0)$. We claim that $g(w) \in \partial B_1(0)$. Indeed, otherwise the continuity of f^{-1} on $B_1(0)$ implies that

$$\partial D_1^n(0) \ni \lim_{j\to\infty}(z'_j, w) = \lim_{j\to\infty} f^{-1}(f(z'_j, w)) = f^{-1}(g(w)) \in D_1^n(0),$$

which gives a contradiction since $D_1^n(0)$ is open. Hence $g(D_1^1(0)) \subset \partial B_1(0)$. By Lemma 9.11 we conclude that g is constant. Hence Theorem 1.5 implies that

$$0 = g'(w) = \lim_{j\to\infty} f_j'(w) = \lim_{j\to\infty} F_w(z_j').$$

Since the sequence $\{z_j'\}_{j\in\mathbb{N}}$ was arbitrary (though recall that we passed to a subsequence), it follows that F_w can be extended continuously to $\partial D_1^{n-1}(0)$ taking the value 0. Applying the maximum principle to each coordinate of F_w we deduce that $F_w \equiv 0$. By the definition of F_w we conclude that $\det(Df(z', w)) = 0$. However, by the chain rule

$$\mathrm{Id} = D(f^{-1})(f(z', w)) \cdot Df(z', w),$$

so that $Df(z', w)$ is invertible. This yields a contradiction.

□

As a final result, we will prove Hartogs's theorem on separate holomorphy. For the proof we will need some results on subharmonic functions and some basic facts on semicontinuous functions.

Definition 9.14. Let X be a metric space. A function $u : X \to \mathbb{R}\cup\{\pm\infty\}$ is called lower semicontinuous if for every $x \in X$ and every sequence $x_n \to x$ it holds that

$$u(x) \le \liminf_{n\to\infty} u(x_n).$$

It is called upper semicontinuous if for every $x \in X$ and every sequence $x_n \to x$ it holds that

$$u(x) \ge \limsup_{n\to\infty} u(x_n).$$

We will use the following elementary properties of lower semicontinuous functions.

Lemma 9.15. *Let X be a metric space and I be a set of indices. If for all $i \in I$ the function $u_i : X \to \mathbb{R}\cup\{\pm\infty\}$ is lower semicontinuous, then the function $\bar{u}(x) := \sup_{i\in I} u_i(x)$ is lower semicontinuous. Moreover, a function $u : X \to \mathbb{R}\cup\{\pm\infty\}$ is lower semicontinuous if and only if the set $\{x \in X : u(x) \le t\}$ is closed for all $t \in \mathbb{R}$*[1].

Proof. Let $x \in X$ and consider a sequence $x_n \to x$. Then

$$\bar{u}(x) = \sup_{i\in I} u_i(x) \le \sup_{i\in I} \liminf_{n\to\infty} \underbrace{u_i(x_n)}_{\le \bar{u}(x_n)} \le \sup_{i\in I} \liminf_{n\to\infty} \bar{u}(x_n) = \liminf_{n\to\infty} \bar{u}(x_n).$$

Hence $\bar{u}$ is lower semicontinuous.

To prove the second assertion, assume first that u is lower semicontinuous and that $x_n \in \{x \in X : u(x) \le t\}$ for all $n \in \mathbb{N}$ with $x_n \to x$ for some $x \in X$. Then by the lower semicontinuity of u we have that

$$u(x) \le \liminf_{n\to\infty} \underbrace{u(x_n)}_{\le t} \le t.$$

Hence $x \in \{x \in X : u(x) \le t\}$.

Conversely, assume that the latter set is closed for all $t \in \mathbb{R}$. Fix $x_0 \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $x_n \to x_0$. If $\liminf_{n\to\infty} u(x_n) = \infty$, then there is nothing to prove. Hence assume that the lim inf is not ∞. Passing to a subsequence realizing the lim inf, we can assume that this lim inf is actually a limit, which we denote by $t_0 \in \mathbb{R}\cup\{-\infty\}$. Fix any $t > t_0$. Then for all sufficiently large $n \in \mathbb{N}$ we have that $x_n \in \{x \in X : u(x) \le t\}$. Since this set is closed by assumption, it follows that $u(x_0) \le t$. This holds for any $t > t_0$, so $u(x_0) \le t_0$. Thus u is lower semicontinuous.

□

Next we define subharmonic functions.

[1]In general topological spaces the closedness of sublevel sets can be taken as the definition of lower semicontinuity.

Definition 9.16. Let $U \subset \mathbb{C}$ be open and $u : U \to \mathbb{R} \cup \{-\infty\}$. We say that u is subharmonic in U if u is upper semicontinuous and for all $z_0 \in U$ and $r > 0$ such that $\overline{B_r(z_0)} \subset U$ we have[2]

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta. \tag{14}$$

Remark 9.17. Note that the integral in (14) has a well-defined value in $\mathbb{R} \cup \{-\infty\}$, since upper semicontinuity of u implies that it is bounded from above on compact sets. Indeed, the sets $\{z \in U : u(z) < t\}$ for $t \in \mathbb{R}$ form an open cover of U (arguing as in Lemma 9.15), and any compact set has a finite subcover.

Remark 9.18. An important class of a subharmonic functions is given by the following example: if $g : U \to \mathbb{C}$ is holomorphic, then

$$u(z) = \begin{cases} \log(|g(z)|) & \text{if } g(z) \neq 0, \\ -\infty & \text{if } g(z) = 0, \end{cases}$$

is subharmonic. Indeed, such a function is clearly upper semicontinuous. Moreover, if z_0 is such that $g(z_0) = 0$, then (14) obviously holds. If $g(z_0) \neq 0$, then (14) follows from Jensen's formula (Theorem 5.1), since it actually holds even if g has zeros in $\partial B_r(z_0)$. This can be seen either by considering nearby circles which avoid zeros and using continuity in r of the sum and integral terms in the formula, or by arguing directly via the identity[3] $\int_0^{2\pi} \log|1 - e^{i\theta}|\, d\theta = 2\int_0^{\pi} \log(2\sin\theta)\, d\theta = 0$ in the last step of the proof.

We will use the following compactness result for sequences of subharmonic functions as a key step in the proof of Hartog's theorem on holomorphy.

Lemma 9.19 (Hartogs's lemma)**.** *Let $u_n : U \to \mathbb{R} \cup \{-\infty\}$ be a sequence of subharmonic functions for which there exist $M > 0$ and $C \in \mathbb{R}$ such that*

$$\begin{aligned} u_n(z) &\le M && \text{for all } n \in \mathbb{N} \text{ and } z \in U, \\ \limsup_{n\to\infty} u_n(z) &\le C && \text{for all } z \in U. \end{aligned}$$

Then for every compact set $K \subset U$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have

$$\sup_{z \in K} u_n(z) \le C + \varepsilon.$$

Proof. For ease of notation, if $z = x + iy$ denote by $dV(z) = dx\, dy$ the volume measure on $\mathbb{C} \cong \mathbb{R}^2$. Recall that if $z = re^{i\theta}$, then in polar coordinates $dV(z) = r\, dr\, d\theta$.

Fix a compact set $K \subset U$ and $0 < \varepsilon < \frac{1}{100}$. Shifting the u_n by an additive constant, we can assume without loss of generality that $C, M \le -1$, so in particular the u_n are all negative. For any $0 < r < \frac{\operatorname{dist}(K, \partial U)}{2}$ and $z_0 \in K$, Fatou's lemma (a corollary of the monotone convergence theorem) and negativity of the u_n give

$$\limsup_{n\to\infty} \int_{B_r(z_0)} u_n(z)\, dV(z) \le \int_{B_r(z_0)} \limsup_{n\to\infty} u_n(z)\, dV(z),$$

so

$$\int_{B_r(z_0)} u_n(z)\, dV(z) \le \pi r^2 (C + \varepsilon) \qquad \text{for all } n \ge n(z_0).$$

Now (recalling $C \le -1$ and $0 < \varepsilon < \frac{1}{100}$) set

$$0 < \delta := r \cdot \left(\sqrt{\frac{C + \varepsilon}{C + 2\varepsilon}} - 1 \right) < r.$$

[2]One can show that this global definition of subharmonic functions is equivalent to the local requirement that (14) holds for all $0 < r < \delta$, for some $\delta > 0$ depending on z_0. See [4, Corollary 2.1.8].

[3]Indeed, $I := \int_0^{\pi} \log(2\sin\theta)\, d\theta = 2\int_0^{\frac{\pi}{2}} \log(2\sin\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \log(4\sin\theta\cos\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \log(2\sin(2\theta))\, d\theta = I/2$.

Then averaging the mean-value inequality (14) over the radius (weighted by the non-negative measure $2\pi r\,dr$), if $|z_0' - z_0| < \delta$ and $n \geq n(z_0)$ then we obtain from the negativity of u_n that

$$\pi(r+\delta)^2 \cdot u_n(z_0') \leq \int_{B_{r+\delta}(z_0')} u_n(z)\,dV(z) \leq \int_{B_r(z_0)} u_n(z)\,dV(z) \leq \pi r^2 (C+\varepsilon).$$

Therefore $u_n(z_0') \leq C + 2\varepsilon$ for all $z_0' \in B_\delta(z_0)$ and $n \geq n(z_0)$. Covering the compact set K with finitely many balls $B_\delta(z_0)$ finishes the proof of the lemma.

□

Theorem 9.20 (Hartogs's theorem on holomorphy)**.** *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be holomorphic separately in each variable. Then f is holomorphic in the sense of Definition 9.1.*

Proof. We shall show that f is locally given by an absolutely converging power series, i.e. that for all $z_0 \in U$ there exists $r > 0$ such that

$$f(z) = \sum_{\alpha \in (\mathbb{N}_0)^n} c_\alpha (z - z_0)^\alpha \qquad \text{for all } z \in D_r^n(z_0). \tag{15}$$

The claim then follows by general properties of converging power series: one first shows that all partial derivatives exist and are continuous. Then differentiability follows by a standard real analysis argument, and to conclude $\mathbb{C}$-linearity of $Df(z_0)$ we use the Cauchy-Riemann equations in each variable (implied by separate holomorphy) as in Exercise H 13.1.

We split the proof of (15) into several steps. Without loss of generality we can assume that $z_0 = 0$ and choose $r > 0$ such that $\overline{D_{2r}^n(0)} \subset U$.

Step 1: Conclude assuming local boundedness.
Here we assume that f is bounded on $\overline{D_r^n(0)}$. Iterating the single-variable Cauchy integral formula, the separate holomorphy of f implies that for all $z \in D_r^n(0)$ we have

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1| = r} \cdots \int_{|\xi_n| = r} \frac{f(\xi)}{\prod_{j=1}^n (\xi_j - z_j)}\, d\xi_n \ldots d\xi_1.$$

For $|z_j| < |\xi_j|$, using the geometric series formula we have that

$$\frac{1}{\xi_j - z_j} = \sum_{n_j = 0}^\infty \frac{z_j^{n_j}}{\xi_j^{n_j + 1}}$$

and the sum converges locally normally for $|\xi_j| = r$ and $|z_j| < r$. Using the boundedness of f, the local normal convergence of the geometric series allows us to switch sums and integrals (Fubini) to obtain

$$f(z) = \sum_{\alpha \in (\mathbb{N}_0)^n} c_\alpha z^\alpha \quad \text{ for } \quad c_\alpha = \frac{1}{(2\pi i)^n} \int_{|\xi_1| = r} \cdots \int_{|\xi_n| = r} \frac{f(\xi)}{\xi^{\alpha + (1,\ldots,1)}}\, d\xi_n \ldots d\xi_1,$$

and the series converges locally normally on $D_r^n(0)$. This is the claimed power series representation, and we conclude that f is holomorphic in the sense of Definition 9.1.

We now prove the general case by induction on the number of variables n. For $n = 1$ the statement is clear. Next, assuming that f is jointly holomorphic in the first $n-1$ variables and also holomorphic in the last variable, we will show that f is locally bounded, hence jointly holomorphic in n variables by Step 1.

Step 2: Local boundedness in a smaller polydisc via the Baire category theorem.
We claim that there exist closed balls with non-empty interior $E_j \subset \overline{B_r(0)} \subset \mathbb{C}$ for $1 \leq j \leq n-1$ and $E_n = \overline{B_r(0)}$ such that f is bounded on $E_1 \times \ldots \times E_n$. (Note, however, that in general $0 \notin E_j$, so this step is not sufficient for our goal of proving complex differentiability at the origin.)

Given $N \in \mathbb{N}$ we define the sets

$$\Omega_N := \left\{ z' \in \prod_{j=1}^{n-1} \overline{B_r(0)} : \sup_{z_n \in \overline{B_r(0)}} |f(z', z_n)| \leq N \right\} \subset \mathbb{C}^{n-1}.$$

Using the induction hypothesis, we know that the function $z' \mapsto f(z', z_n)$ is in particular continuous, so that by Lemma 9.15 the function $z' \mapsto \sup_{z_n \in \overline{B_r(0)}} |f(z', z_n)|$ is lower semicontinuous. Hence, again by Lemma 9.15, the set Ω_N is closed. Moreover, recalling that $\overline{D^n_{2r}(0)} \subset U$, for any fixed $z' \in \prod_{j=1}^{n-1} \overline{B_r(0)}$ the function $z_n \to f(z', z_n)$ is holomorphic on $B_{2r}(0)$ and thus bounded on $\overline{B_r(0)}$. It follows that

$$\bigcup_{N\in\mathbb{N}} \Omega_N = \prod_{j=1}^{n-1} \overline{B_r(0)}.$$

The set on the right-hand side has non-empty interior, so that by the Baire category theorem there exists $N \in \mathbb{N}$ such that Ω_N has non-empty interior. In particular, this set Ω_N contains a closed polydisc with non-empty interior. The definition of the set Ω_N yields the claim of Step 2.

Step 3: From boundedness on smaller to boundedness on larger polydiscs via subharmonic functions. We show that if $f : D^n_r(z_0) \to \mathbb{C}$ is separately holomorphic in z' and z_n, and bounded on a smaller polydisc $D^n_{(r',\dots,r',r)}(z_0)$ with $r' < r$, then it is holomorphic on $D^n_r(z_0)$. (Note that the center of the two polydiscs is the same, in contrast to what we obtained in Step 2.)

To reduce notation, we once again assume without loss of generality that $z_0 = 0$. By holomorphy in z', for $(z', z_n) \in D^n_r(0)$ we can write

$$f(z', z_n) = \sum_{\alpha \in (\mathbb{N}_0)^{n-1}} c_\alpha(z_n)(z')^\alpha, \tag{16}$$

where the series converges locally normally with respect to $z' \in D^{n-1}_r(0)$ and by Corollary 9.6 (*ii*) the coefficients are given by

$$c_\alpha(z_n) = \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial (z')^\alpha}(0, z_n).$$

Again due to the holomorphy in z', we can use Cauchy's integral formula for the derivative, given in Corollary 9.6 (*i*), to obtain

$$c_\alpha(z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{|\xi_1| = \frac{r'}{2}} \cdots \int_{|\xi_{n-1}| = \frac{r'}{2}} \frac{f(\xi_1, \dots, \xi_{n-1}, z_n)}{\xi_1^{\alpha_1+1} \cdots \xi_{n-1}^{\alpha_{n-1}+1}} \, d\xi_{n-1} \dots d\xi_1. \tag{17}$$

Since f is bounded and thus jointly holomorphic on $D^n_{(r',\dots,r',r)}(0)$, it follows (from joint continuity of f and holomorphicity in z_n) that $c_\alpha(z_n)$ is holomorphic for $z_n \in B_r(0) \subset \mathbb{C}$. Hence $v_\alpha(z_n) = \frac{1}{|\alpha|} \log(|c_\alpha(z_n)|)$ defines a family of subharmonic functions on $B_r(0)$, for $\alpha \neq \bar{0}$.

We next verify the assumptions of Lemma 9.19, implicitly numbering the countably many multi-indices $\bar{0} \neq \alpha \in (\mathbb{N}_0)^{n-1}$, so that v_α can be seen as a sequence. First, note that since the sum in (16) converges locally normally with respect to z', it follows that for any $0 < r_2 < r$ we have

$$\lim_{|\alpha| \to \infty} |c_\alpha(z_n)| \cdot r_2^{|\alpha|} = 0 \quad \text{for all } z_n \in B_r(0),$$

which implies

$$\limsup_{|\alpha| \to \infty} v_\alpha(z_n) \leq -\log(r_2).$$

Moreover, from (17) and the standard estimate for contour integrals we deduce that

$$|c_\alpha(z_n)| \leq \frac{\sup_{z' \in \partial D^{n-1}_{r'/2}(0)} |f(z', z_n)|}{\left(\frac{r'}{2}\right)^{|\alpha|}} \leq \frac{B}{\left(\frac{r'}{2}\right)^{|\alpha|}},$$

where $B \geq 1$ is a bound for $|f|$ on the polydisc $D^n_{(r',\dots,r',r)}(0)$. Taking the logarithm, we find that for all $z_n \in B_r(0)$ it holds that

$$v_\alpha(z_n) \leq \frac{1}{|\alpha|} \log(B) - \log(\tfrac{r'}{2}) \leq \log(B) - \log(\tfrac{r'}{2}).$$

Hence v_α is uniformly bounded from above and we can apply Lemma 9.19 to deduce that for all $0 < r_1 < r_2$ there exists $N \in \mathbb{N}$ such that for all $|\alpha| \geq N$ and all $z_n \in B_{r_1}(0)$ it holds that $v_\alpha(z_n) \leq -\log(r_1)$, which is equivalent to

$$|c_\alpha(z_n)| \cdot r_1^{|\alpha|} \leq 1 \quad \text{for all } z_n \in B_{r_1}(0).$$

From Exercise H 13.4 b) we thus infer that the series

$$f(z', z_n) = \sum_{\alpha \in (\mathbb{N}_0)^{n-1}} c_\alpha(z_n)(z')^\alpha$$

converges uniformly on $\overline{D^n_{r_0}(0)}$ for all $0 < r_0 < r_1$. In particular it is bounded on this set and therefore jointly holomorphic on its interior, by Step 1. Since the radii $0 < r_0 < r_1 < r_2 < r$ were arbitrary, we deduce that f is holomorphic on $D^n_r(0)$.

Step 4: Geometric conclusion.

By Step 2 we find a closed polydisc $\overline{D^{n-1}_{r'}(z_0)} \times \overline{B_r(0)}$ on which f is bounded. In addition, we know that $(z_0, 0) \in D^n_r(0)$ since this closed polydisc is a subset of $\overline{D^n_r(0)}$. By Step 3 we know that f is holomorphic on $D^{n-1}_r(z_0) \times B_r(0)$, so the condition $z_0 \in D^{n-1}_r(0)$ yields that $(0', 0) \in D^{n-1}_r(z_0) \times B_r(0)$. Thus f is holomorphic at the origin, and this concludes the proof.

□

References

[1] A. F. Beardon and D. Minda, On the pointwise limit of complex analytic functions, *Amer. Math. Monthly*, *110(4)*, 2003, 289–297.

[2] S. G. Krantz, *Function theory of several complex variables*, Pure and applied mathematics, Wiley 1982.

[3] C. Laurent-Thiébaut, *Holomorphic function theory in several variables: an introduction*, Springer London, 2011.

[4] R. Remmert, *Classical topics in complex function theory*, Graduate texts in mathematics 172, transl. by L. Kay, Springer New York, 1998.

[5] E. M. Stein and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, 2, Princeton Univ. Press, 2003.

[6] E. M. Stein and R. Shakarchi, *Functional analysis*, Princeton Lectures in Analysis, 4, Princeton Univ. Press, 2011.

[7] V. Scheidemann, *Introduction to complex analysis in several variables*, Birkhäuser Basel, 2005.